Confounded Observational Learning with Common Values

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Abstract

We analyze observational learning when a fraction of players are uninformed of the actions of previous agents, and act based only on their private information, while informed players observe all previous actions. Informed players are uncertain about the true proportion of uninformed players. They simultaneously learn about this proportion and about the payoff-relevant state. Confounded learning emerges as a robust phenomenon in this environment. It is also globally stable for a generic set of parameter values, so that public beliefs converge to the confounded learning point with positive probability, starting from almost all current beliefs. We also show that correct learning is always globally stable. In contrast, correct learning may not be globally stable when it arises due to heterogeneous preferences as in Smith and Sørensen (2000).

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1 Introduction

The seminal papers of Banerjee (1992) and Bikhchandani et al. (1992) established the possibility of herd behavior and information cascades. These papers analyze Bayesian players, who receive boundedly informative private signals, and learn from the actions of previous actors. When incorrect herding happens, social learning stops; all but a finite number of players end up choosing the wrong action, even though society could learn the correct state if it were able to aggregate the information available to individuals. The possibility of incorrect herding depends crucially upon the private signals being boundedly informative. Smith and Sørensen (2000) show that complete learning is guaranteed, if players have common preferences, and their private signals are of unbounded strength. They show that learning must necessarily be complete, i.e. the public belief must assign probability one to the true state in the long run.

In this paper, we examine the implications of having a fraction of "uninformed" players, who do not observe the actions of their predecessors. Alternatively, these players are behavioral, and ignore past actions, an assumption that is in line with experimental evidence – see Duffy et al. (2016),Weizsäcker (2010) and Ziegelmeyer et al. (2013).[1] Provided that the prior belief is not extreme, the presence of these uninformed players can potentially play the same role as the assumption of unbounded signals, by ensuring that players' decisions always contain an amount of information that is bounded away from zero. Since each player observed could be uninformed, his action statistically reveals his private information.

Following this intuition, it is straightforward to show that complete learning is guaranteed for the rational players provided that the rational players know the precise proportion of uninformed players, even when private signals are of bounded strength. However, it may be unrealistic to assume that the rational players know the precise proportion of uninformed players. This leads us to consider a model with higher-dimensional uncertainty – rational players are uncertain both about the payoff relevant state, and about the proportion of uninformed players, and will learn about both aspects as the game progresses. Our main finding is that complete learning is possible, but it is not guaranteed. In the long run, learning could be confounded, with the society's limit beliefs assigning positive weight both to the true state – which is two-dimensional – and its "opposite", i.e. the state that is incorrect on both dimensions. That is, if the true payoff relevant state is A and the proportion of uninformed players is L (for low), society can assign positive probability to the pair (A, L) and to the

[1] These experiments find that there exist individuals who decide exclusively based on their own private information, ignoring prior actions.

pair (B, H). Since beliefs about the payoff relevant state are interior at the confounded learning point, in the long run each rational player still uses his private information to decide. Notably, confounded learning arises even though all players have common values, i.e. identical preferences over state-action pairs.[2]

The message of the previous paragraph can be restated more precisely as follows: our model shows that there exist multiple stationary points of the stochastic process of public beliefs — the complete learning point, and the confounded learning point. This raises additional questions. Does either of these points arise for a distant prior belief? In other words, is either of these points *globally stable* – does the belief process converges to a stationary point with positive probability, starting from *any* current posterior belief, for a large set of initial priors that allow learning?

Our answers to these questions, in the context of our model, are:

- Complete learning is always globally stable.
- Confounded learning is globally stable for a generic set of parameters.

This is in sharp contrast with existing literature, where the existence of confounded learning could preclude complete learning for a large set of reasonable priors.

The paper is organized as follows. We first discuss the related literature. Section 2 sets out the model. Section 3 analyzes the evolution of society's posterior beliefs along the equilibrium path. In section 4 we explain the intuition for confounded learning, and provide necessary and sufficient conditions for confounded learning to arise. In section 5 we establish that complete learning is globally stable in our model. Section 6 shows that confounded learning could be globally stable.

1.1 Related Literature

There is an extensive literature on observational learning. In this section, we focus our attention on the papers that are most closely related.

Smith and Sørensen (2000) (SS henceforth) provide a comprehensive analysis of observational learning, and also developed many of the technical insights that underlie the analysis in the present paper. They were also the first to show that confounded learning is possible when players have divergent preferences. [3] In SS, a fraction of players would like to choose

[2]Smith and Sørensen (2000) show that confounded learning is possible when players do not have common values. We discuss their work more fully later.

[3]Easley and Kiefer (1988) examine individual learning (rather than social learning) and find that confounded learning is possible, for non-generic parameters.

their action to match the state, while the remaining fraction prefer to mismatch action and state. In our paper, the underlying economic environment that gives rise to confounded learning is very different. Players have common values, and every player would like her action to match the state. Since players do not also know the true proportion of uninformed players, uncertainty is two-dimensional in our model, while it is one-dimensional in SS. Our substantive results also differ. In SS, confounded learning could preclude the possibility of complete learning. In our model, complete learning must happen with strictly positive probability and is globally stable.

Bohren (2016) allows for uninformed players, and assumes that rational players have a wrong but fixed belief about the proportion of uninformed players. She finds that if the belief is not too wrong, complete learning is guaranteed, but for a large error, the posterior-belief process may eventually assign probability zero to the true state or fail to converge. Bohren and Hauser (2018) generalize this work by allowing more channels of mis-specifications. They allow each player misinterpret private signal and to hold wrong belief of the true distribution of players types. An uninformed player in Bohren (2016) is a player who correctly interprets the private signal but mistakenly think other players all acts noisily, and a (biased) rational player in Bohren (2016) is a player who correctly interprets the private signal but holds a fixed wrong belief about the proportion of uninformed players. If all types have an approximately correct interpretation of signals and an approximately correct belief regarding the distribution of other players' types, then complete learning is ensured. Otherwise, several pathological outcomes could arise: (1) it is possible that some types assign probability 1 to one state while other types assign probability 1 to the other state (asymptotic disagreement); (2) it is possible that some types' long run posterior beliefs settle down while other types' posterior beliefs keep cycling; (3) It is also possible that all types assign all the weight to the wrong state (asymptotic mislearning). Our results are very different from above two papers — there cannot be incorrect learning or cyclical beliefs, and there can be confounded learning. These differences arise since rational players use history to revise their beliefs on the true proportion of uninformed players. In other words, players in our model are uncertain, but their beliefs are correctly specified.

Bohren's unpublished Ph.D. thesis (Bohren (2012)) also examines a model where rational players learn the true proportion of uninformed players. She provides an example with binary signals that displays confounded learning. This example is non-generic – for confounded learning to emerge, the parameters of the model must satisfy a single equation. Consequently, confounded learning disappears if the primitives of the model are slightly perturbed. In our model, with a continuum of signals, confounded learning emerges for an open set of parameters, and is hence robust to small perturbations in parameter values. We also believe that the results presented here constitute a more systematic and comprehensive analysis of the problem. In particular, we examine the local and global stability of the stationary points.

Wolitzky (2018) studies a deterministic social learning model in which confounded learning and complete learning coexists. One of his finding is that the complete learning can never be reached if the complete learning point is separated from the starting point of learning by a line representing confounded learning. In other words, complete learning in his model fails to be globally stable. In contrast, complete learning in our model is always globally stable. This is because the uncertainty in our model is of two-dimensional and hence the confounded learning point can never separate the complete learning point and any current posterior belief.

Other related literature include Eyster and Rabin (2010) and Acemoglu et al. (2010). Eyster and Rabin (2010) assumes every player is rational but mistakenly think other players are uninformed. They find incorrect herding could happen even with continuum actions and unbounded signals. Acemoglu et al. (2010) assumes two types of players who differ in their preferences. Confounded learning arises when preferences are sufficiently heterogeneous.

2 Model

The model is an infinite horizon, discrete-time model. There is a two-dimension uncertainty: payoff-relevant states $\Omega_1 = \{A, B\}$ and proportions of uninformed players $\Omega_2 = \{L, H\}$. For abbreviation, we shall refer $\omega_1 \in \Omega_1$ as "payoff state", and $\omega_2 \in \Omega_2$ as "type state".

In period 0, nature chooses one state out of four potential states

$$\Omega = \Omega_1 \times \Omega_2 = \{AL, AH, BL, BH\}$$

according to a common prior $\Lambda_0 = (\lambda_0^{AH}, \lambda_0^{BL}, \lambda_0^{BH})$, which assigns positive weight to all four states. Throughout this paper, a belief over the state space Ω is written as three ratios with the probability associated with state AL in the denominator. For example: $\lambda_0^{AH} = \frac{\Pr(AH|\emptyset)}{\Pr(AL|\emptyset)}$.

In each period $t \geq 1$, one player arrives. He chooses between actions $\{a, b\}$ with the objective to match the realized payoff state. The utility function $u : \{a, b\} \times \Omega_1 \to \{0, 1\}$ is identical for every player and is given as

$$u(a, A) = (b, B) = 1; \qquad u(a, B) = u(b, A) = 0. \tag{1}$$

As standard in the literature, one player's payoff depends only on his action and the realized payoff state, and is independent from other players' actions.

Before taking an action, each player observes a private signal $\mathcal{S}_t$ from a common signal space. The distribution of the private signal depends on the realized payoff state. Following the literature, we identify a player's private signal $\mathcal{S}_t$ with his private belief s_t as if the payoff state is equally likely to be A and B:

$$s_t = \Pr(A|\mathcal{S}_t) = \frac{\Pr(\mathcal{S}_t|A)\frac{1}{2}}{\Pr(\mathcal{S}_t|A)\frac{1}{2} + \Pr(\mathcal{S}_t|B)\frac{1}{2}}. \tag{2}$$

In other words, the private belief s_t of player t is the probability attached to payoff state being A conditional solely on the private signal $\mathcal{S}_t$. The distribution of s_t is denoted as $F^{\omega_1}(s)$ with $\omega_1 \in \{A, B\}$. We assume $\mathcal{S}_t$ is i.i.d across players, and hence so is s_t. We further introduce the following assumption:

Assumption 1 *$F^A(s)$ and $F^B(s)$ are mutually absolutely continuous, non-atomic, and have common support as*

$$supp(F^A(s)) = supp(F^B(s)) = (\underline{s}, \overline{s}) \subset (0, 1),$$

where $\underline{s} < \frac{1}{2} < \overline{s}$. $F^A(s), F^B(s)$ are twice continuously differentiable on $(\underline{s},\overline{s})$.

The prior belief is not so extreme that uninformed players always choose one action:

$$\underline{s} < \frac{\lambda_0^{BH} + \lambda_0^{BL}}{1 + \lambda_0^{AH} + \lambda_0^{BL} + \lambda_0^{BH}} < \overline{s}. \tag{3}$$

Note that here we do not make an assumption on the strength of private signals. All the arguments apply to both bounded and unbounded private signals, provided that condition 3 is satisfied.

Rational players also observe the public history of previous actions. If player t is rational, then he observes $h_t = (a_1, \ldots, a_{t-1})$, i.e the sequence of actions taken in previous periods. uninformed players do not observe any previous actions. The realization of each player to be uninformed is i.i.d across players. The probability that any player is uninformed is either p_L or p_H, depending on the realized type state.

3 The Process of Learning

Our analysis focuses on the posterior belief over the state space Ω conditional on a realized public history h_t. Specially, we ask whether the society's posterior beliefs settle down to a limit belief, and whether this limit belief assigns all the weight to the realized state. Following the literature, we say "the society learns" if the posterior beliefs settle down to a limit belief. Furthermore, we say that "learning is complete" if the limit belief assigns all the weight to the realized state $\omega \in \Omega$. Complete learning guarantees information aggregation, and is of particular interest.

In this section, we study how posterior belief evolves from period t to period $t+1$. We conclude that posterior beliefs always settle down as a result of martingale property. In other words, society always learns.

First, we solve for the unique sequential equilibrium. Without loss of generality, from now on we assume the realized state is AL. We introduce the following notation. Player t's information set is denoted as $I_t = \{s_t, PI_t\}$, where PI is an abbreviation used for "public information". If player t is rational, then $PI_t = h_t$; if player t is uninformed, then $PI_t = \emptyset$. Player t's strategy σ_t is a function from I_t to a distribution over actions $\{a, b\}$. For each $\omega \in \Omega$, strategies $\sigma_1, \dots, \sigma_t$ determines the probability of each history $h_{t+1} \in \{a, b\}^t$. We use $\mathbb{P}_t$ to denote the probability measure induced on $\mathcal{H}_t = \Omega \times \{a, b\}^t$, with the understanding that $\mathbb{P}_t$ actually depends on some strategy profile. [4] Strategies $\sigma = \{\sigma_1, \dots\}$ form an equilibrium if $\forall t$

$$\sigma_t(I_t) = \begin{cases} a, & \text{if } \frac{\mathbb{P}_{t-1}(B|PI_t)}{\mathbb{P}_{t-1}(A|PI_t)} \frac{1-s_t}{s_t} \leq 1; \\ b, & \text{if } \frac{\mathbb{P}_{t-1}(B|PI_t)}{\mathbb{P}_{t-1}(A|PI_t)} \frac{1-s_t}{s_t} \geq 1. \end{cases} \tag{4}$$

This definition is actually quite intuitive. Because public information PI_t is independent of private belief s_t, $\frac{\mathbb{P}_{t-1}(B|PI_t)}{\mathbb{P}_{t-1}(A|PI_t)} \frac{1-s_t}{s_t}$ actually represents the posterior likelihood ratio of payoff state being B over being A conditional on player t's information set I_t. Therefore, definition 4 says σ is an equilibrium if player t choose the action matching the more plausible payoff state conditional on his information set.

One immediate observation from definition 4 is that player t's equilibrium strategy can be represented as a cutoff rule in terms of his private belief s_t.

[4] Here $\mathbb{P}_t(\omega \times h_{t+1}) > 0$ for all $\omega \in \Omega$ and $h_{t+1} \in \{a, b\}^t$, since uninformed players exist.

Lemma 2 *Up to a tie-breaking rule, the unique equilibrium is given as*

$$\sigma_t = a \Leftrightarrow \begin{cases} s_t \geq \frac{\lambda_0^{BH}+\lambda_0^{BL}}{\lambda_0^{BH}+\lambda_0^{BL}+\lambda_0^{AH}+1}, \text{ if player } t \text{ is uninformed;} \\ s_t \geq \mathbb{P}_{t-1}(B|h_t), \text{ if player } t \text{ is rational.} \end{cases} \tag{5}$$

In the above lemma, we assume action a is chosen when the player think two payoff states are equally plausible. This tie-breaking rule is immaterial, since the probability of a tie is zero due to continuous private belief.

From now on, we use σ to denote the equilibrium given in Lemma 2, use $\mathbb{P}_t$ to represent the probability measure on $\mathcal{H}_t$ induced by the equilibrium, and use $\mathbb{P}$ to represent the probability measure on $\mathcal{H} = \Omega \times \{a, b\}^{\mathbb{N}}$ induced by the equilibrium. When we talk about the posterior belief conditional on h_t, it is the posterior belief with respect to $\mathbb{P}_{t-1}$. Since there are four potential states $\{AL, AH, BL, BH\}$, we can summarize society's posterior belief at period t as a random vector of three likelihood ratios. With probability associated with the true state AL in the denominator, we write the posterior belief Λ_t as

$$\Lambda_t \equiv (\lambda_t^{AH}, \lambda_t^{BL}, \lambda_t^{BH}) \equiv \Big(\frac{\mathbb{P}_{t-1}(AH|h_t)}{\mathbb{P}_{t-1}(AL|h_t)}, \frac{\mathbb{P}_{t-1}(BL|h_t)}{\mathbb{P}_{t-1}(AL|h_t)}, \frac{\mathbb{P}_{t-1}(BH|h_t)}{\mathbb{P}_{t-1}(AL|h_t)}\Big). \tag{6}$$

We denote the equilibrium probability of $\sigma_t = \alpha, \forall \alpha \in \{a, b\}$ at state $\omega_1\omega_2$ with belief Λ_t as $\phi(\alpha|\omega_1\omega_2, \Lambda_t)$. To represent the equilibrium probability, it is convenient to introduce random variable $x_t(\Lambda_t)$ for a belief $\Lambda_t = (\lambda_t^{AH}, \lambda_t^{BL}, \lambda_t^{BH})$ as

$$x_t(\Lambda_t) = \frac{\lambda_t^{BH} + \lambda_t^{BL}}{1 + \lambda_t^{AH} + \lambda_t^{BL} + \lambda_t^{BH}}. \tag{7}$$

We can verify that $x_t(\Lambda_t(h_t)) = \mathbb{P}_{t-1}(B|h_t)$. Then we have

$$\phi(\alpha|\omega_1\omega_2, \Lambda_t) = \phi(\alpha|\omega_1\omega_2, x_t) = \begin{cases} p_{\omega_2}(1 - F^{\omega_1}(x_0)) + (1 - p_{\omega_2})(1 - F^{\omega_1}(x_t)), \text{ if } \alpha = a; \\ p_{\omega_2}F^{\omega_1}(x_0) + (1 - p_{\omega_2})F^{\omega_1}(x_t), \text{ if } \alpha = b. \end{cases}$$

Here and from now on, we use $x_0 = \frac{\lambda_0^{BH}+\lambda_0^{BL}}{1+\lambda_0^{AH}+\lambda_0^{BL}+\lambda_0^{BH}}$ to represent the probability assigned to payoff state being B at prior belief $(\lambda_0^{AH}, \lambda_0^{BL}, \lambda_0^{BH})$.

With posterior belief Λ_t defined, we can state the definitions of learning rigorously.

Definition 3 *Given a history $h \in \{AL\} \times \{a,b\}^{\mathbb{N}}$, the society learns along h if*

$$t \to +\infty \Rightarrow (\lambda_t^{AH}(h), \lambda_t^{BL}(h), \lambda_t^{BH}(h)) \text{ converges}$$

and learning is complete along h if

$$(\lambda_t^{AH}(h), \lambda_t^{BL}(h), \lambda_t^{BH}(h)) \to (0,0,0).$$

At the beginning of this section, we vaguely state that the society learns if posterior beliefs settle down. Here "settling down" is rigorously defined using the notion of convergence. Furthermore, since in $\Lambda_t(h)$ the posterior probability associated with realized state AL is in the denominator, $\Lambda_t(h) \to (0,0,0)$ means that all the weight is assigned to AL.

The following lemma shows that $\lambda_t^{\omega_1\omega_2}$, when restricted on $\{AL\} \times \{a,b\}^{\mathbb{N}}$, forms a non-negative martingale for $\omega_1\omega_2 \in \{AH, BL, BH\}$. The martingale convergence theorem (Theorem 11.5 in Williams (1991)) states that a non-negative martingale almost surely converges to a finite random variable. Therefore, we conclude that almost surely posterior beliefs always settle down to a limit belief along the equilibrium, and the society (almost) always learns.

Lemma 4 *For $\omega_1\omega_2 \in \{AH, BL, BH\}$, $\{\lambda_t^{\omega_1\omega_2}\}_{t\in\mathbb{N}}$ forms a non-negative martingale when restricted to $\{AL\} \times \{a,b\}^{\mathbb{N}}$.*

Proof. See Appendix A. ■

Proposition 5 *There exists a null set $E \subset \{AL\} \times \{a,b\}^{\mathbb{N}}$, such that for any sequence of actions under the realized state $h \in \{AL\} \times \{a,b\}^{\mathbb{N}} - E$, we have*

$$(\lambda_t^{AH}(h), \lambda_t^{BL}(h), \lambda_t^{BH}(h)) \to (\lambda_\infty^{AH}(h), \lambda_\infty^{BL}(h), \lambda_\infty^{BH}(h)), \tag{8}$$

with $\lambda_\infty^{\omega_1\omega_2} < +\infty$, $\omega_1\omega_2 \in \{AH, BL, BH\}$.

In other words, conditional on realized state AL, the posterior belief $(\lambda_t^{AH}, \lambda_t^{BL}, \lambda_t^{BH})$ converges almost surely to a finite random vector.

Proof. This result follows directly from lemma 4 and the martingale convergence theorem (Theorem 11.5 in Williams (1991)). ■

4 Possibility of Confounded Learning

In the previous section, we showed that society's posterior beliefs settle down to a limit belief almost surely. A natural question is whether the limit belief necessarily assigns all the weight to the realized state AL, i.e. whether learning is complete. In this section, we conclude that it is not necessarily the case. If the proportion of uninformed players in H-state is sufficiently higher than in L-state, then it is possible that the limit belief assigns positive weights to both states BH and AL, and 0 weight to states AH and BL. Under such a limit belief, any observed actions happen with equal probability across BH and AL. Therefore, in the limit, even if players still use their private information to decide, their actions stop providing information regards the likelihood ratio of BH and AL. Following Smith and Sørensen (2000), we say "learning is confounded". Confounded learning is very different from information cascade. When learning stops due to an information cascade, the information contained in publicly observed actions overwhelms any player's private signal. As a result, all the players abandon their private signals and herd. However, in confounded learning, the information contained in public actions is inconclusive, and players still use private information to decide.

We can intuitively understand this result in the following way. Since society's posterior beliefs always settle down, the observed action frequency also settles down. Without loss of generality, we can think in terms of the frequency of action b. To have positive weight assigned to state BH, the observed limit frequency of action b must be plausible under BH. When the payoff state is B rather than A, then both types of players are more likely to choose action b. However, the increase of limit frequency of action b due to payoff state change can be balanced by the type state changing from L to H. If the limit belief assigns more weight on payoff state being B than the prior belief does, the rational players, who observe the limit belief, are more likely to choose action b. There are fewer rational players under state BH, hence the limit frequency of action b will move down.

To summarize, if the limit belief assigns more weight to the payoff state being B than the prior belief does, then in state AL, actions b is generally less likely, but there is a high proportion of rational players can counterbalance the effect. In state BH, action b is generally more likely, but there is low proportion of rational players. These two forces can be balanced, provided that there is a sufficient fall in the number of rational players from AL to BH. In fact, this balance is a special case of Simpson's paradox. The probability of action b is strictly higher among rational players and among uninformed players under state BH than under state AL. However, the average probability among all players could be equal across these two states, as long as there is a sufficient change of proportion of uninformed players.

A similar argument shows that the limit belief cannot assign positive weight to AH and BL. In fact, any observed limit frequency of action b is incompatible with state BL. Knowing the limit belief, rational players know the frequency of action b should be higher than observed if the state is BL. See Proposition 7 for an argument of AH.

Above findings generalize the observation in section 1.4 of Bohren (2012). Bohren studies learning with unknown proportion of uninformed players in a special example with symmetric binary private signals. She observes that with proper parameters two different states may be indistinguishable in the long run, for the reason that the probability of observable actions is the same across these two states. Though her observation bears similar characteristic, our findings are more general and insightful. With a symmetric binary signal structure, parameters in her model must satisfy one "equation" to lead to incomplete learning. This means incomplete learning is not a robust phenomenon in her model. With slight perturbation of the parameters, incomplete learning disappears. Our model assumes a continuous private signal structure. The condition of confounded learning is determined by inequality 10. Hence confounded learning is a robust phenomenon in our model. Bohren (2012)'s assumption of symmetric binary signals also simplifies the argument. With proper parameters, the likelihood ratio between these two indistinguishable states stops evolving immediately after herding. In our model, as long as posterior belief at period t doesn't equal the confounded limit belief, all the likelihood ratios still adjust upon observing period t's action. Therefore more dynamics analysis is needed. We shall explore the dynamics property of our model in following sections.

Smith and Sørensen (2000) find confounded learning could arise when players have sufficiently heterogeneous preferences. We remark that our result is quite different from theirs. From the economics perspective, our model assumes all the players have common values, and confounded learning arises because of the unknown proportion of uninformed players.

In the rest of this section, we formalize above intuition of confounded learning's existence. The first observation is due to Smith and Sørensen (2000), and states that society's limit belief must be a stationary point of stochastic process $\Lambda_t = (\lambda_t^{AH}, \lambda_t^{BL}, \lambda_t^{BH})$.

Lemma 6 *Let* $\boldsymbol{\pi} = (\pi_{AH}, \pi_{BL}, \pi_{BH}) \in \mathbb{R}^3$ *satisfying that* $\pi_{\omega_1\omega_2} \geq 0$, $\forall \omega_1\omega_2 \in \{AH, BL, BH\}$. *Let*

$$S = \{h \in \{AL\} \times \{a, b\}^{\mathbb{N}} | (\lambda_\infty^{AH}(h), \lambda_\infty^{BL}(h), \lambda_\infty^{BH}(h)) = (\pi_{AH}, \pi_{BL}, \pi_{BH})\}.$$

If $\mathbb{P}(S) > 0$*, then*

$$\pi_{\omega_1\omega_2} = \pi_{\omega_1\omega_2}\frac{\phi(\alpha|\omega_1\omega_2, \boldsymbol{\pi})}{\phi(\alpha|AL, \boldsymbol{\pi})}, \forall\alpha \in \{a, b\}. \tag{9}$$

In other words, if stochastic process Λ_t *converges to* $(\pi_{AH}, \pi_{BL}, \pi_{BH})$ *with strictly positive probability, then* $(\pi_{AH}, \pi_{BL}, \pi_{BH})$ *must be a stationary point of* Λ_t*.*

Proof. The result follows Theorem B.2 in Smith and Sørensen (2000). ■

Equation 9 says that $\pi_{\omega_1\omega_2} \neq 0$ implies $\phi(\alpha|\omega_1\omega_2, \boldsymbol{\pi}) = \phi(\alpha|AL, \boldsymbol{\pi})$. Intuitively, this means that if limit belief $(\pi_{AH}, \pi_{BL}, \pi_{BH})$ assigns positive weight to state $\omega_1\omega_2$, then limit frequency of action $\alpha \in \{a, b\}$ must be indistinguishable across states $\omega_1\omega_2$ and AL.

Using Lemma 6, we can prove our intuition that the limit belief must assign zero weight to states AH and BL.

Proposition 7 *If stochastic process* $(\lambda_t^{AH}, \lambda_t^{BL}, \lambda_t^{BH})$ *converges to* $(\pi_{AH}, \pi_{BL}, \pi_{BH})$ *with strictly positive probability, then* $\pi_{AH} = \pi_{BL} = 0$*.*

Proof. First, we have

$$\phi(b|BL, x(\boldsymbol{\pi})) = p_L F^B(x_0) + (1 - p_L)F^B(x(\boldsymbol{\pi})),$$

and that

$$\phi(b|AL, x(\boldsymbol{\pi})) = p_L F^A(x_0) + (1 - p_L)F^A(x(\boldsymbol{\pi})).$$

By definition $\frac{f^B(s)}{f^A(s)} = \frac{1-s}{s}$, so $f^B(s) > f^A(s)$ on $(\underline{s}, \frac{1}{2})$ and $f^B(s) < f^A(s)$ on $(\frac{1}{2}, \overline{s})$. Then it follows that

$$\begin{cases} F^B(s) > F^A(s), \text{ if } s \in (\underline{s}, \overline{s}); \\ F^B(s) = F^A(s), \text{ if } s \in [0, \underline{s}] \cup [\overline{s}, 1]. \end{cases}$$

Due to assumption that $x_0 \in (\underline{s}, \overline{s})$, we have $\phi(b|BL, \boldsymbol{\pi}) > \phi(b|AL, \boldsymbol{\pi})$.

Second, we have

$$\phi(b|AH, x(\boldsymbol{\pi})) = p_H F^A(x_0) + (1 - p_H)F^A(x(\boldsymbol{\pi})).$$

To have $\phi(b|AH, \boldsymbol{\pi}) = \phi(b|AL, \boldsymbol{\pi})$, we must have $x(\boldsymbol{\pi}) = x_0$. But if this is the case, then

$$\phi(b|BH, \boldsymbol{\pi}) = F^B(x_0) \neq F^A(x_0) = \phi(b|AL, \boldsymbol{\pi}),$$

which means $\pi_{BH} = 0$. That is, $x(\boldsymbol{\pi}) = x_0$ implies that zero weight must be assigned to state BH. We have shown that zero weight must be assigned to state BL in the first part of this proof. Therefore, $x(\boldsymbol{\pi}) = x_0$ implies that zero weight must be assigned to payoff state being B under belief $\boldsymbol{\pi}$, and this is a contradiction. ■

In the next proposition, we rigorously prove that the limit belief can assign positive weight to state BH if it assigns more weight to payoff state being B. We can also prove that such a limit belief must be unique.

Proposition 8 *If*

$$\phi(b|AL, \bar{s}) > \phi(b|BH, \bar{s}) \tag{10}$$

then there exists unique $\pi^*_{BH} > \max\{\frac{x_0}{1-x_0}, \frac{1-p_H}{1-p_L}\}$ *such that*

$$\phi(b|AL, (0, 0, \pi^*_{BH})) = \phi(b|BH, (0, 0, \pi^*_{BH})) \tag{11}$$

In other words, when condition 10 is satisfied, $\Lambda^* \equiv (0, 0, \pi^*_{BH})$ *gives the unique limit belief where the observed frequency of action b is compatible with state* BH*.*

Proof. Let

$$\begin{aligned} \mathfrak{D}(x) &= \phi(b|BH, x) - \phi(b|AL, x) \\ &= [p_H F^B(x_0) + (1 - p_H)F^B(x)] - [p_L F^A(x_0) + (1 - p_L)F^A(x)] \end{aligned}$$

be defined on $x \in [0, 1]$. Condition 10 is equivalent to that $\mathfrak{D}(\bar{s}) < 0$. Since $F^B(x_0) > F^A(x_0)$, $\mathfrak{D}(x_0) > 0$ and $\mathfrak{D}(\underline{s}) > 0$ always hold. We have

$$\mathfrak{D}'(x) = (1 - p_H)f^B(x) - (1 - p_L)f^A(x).$$

By definition $\frac{f^B(s)}{f^A(s)} = \frac{1-s}{s}$, so $\mathfrak{D}'(x) > 0$ on $(\underline{s}, \frac{1-p_H}{2-p_H-p_L})$ and $\mathfrak{D}'(x) < 0$ on $(\frac{1-p_H}{2-p_H-p_L}, \bar{s})$. Thus, there is an unique $x^* \in (\max\{x_0, \frac{1-p_H}{2-p_H-p_L}\}, \bar{s})$ such that $\mathfrak{D}(x^*) = 0$. Uniqueness of π^*_{BH} follows directly. ■

We conclude this section by stating that long run learning is either complete or confounded.

Proposition 9 *If stochastic process* $(\lambda^{AH}_t, \lambda^{BL}_t, \lambda^{BH}_t)$ *converges to* $(\pi_{AH}, \pi_{BL}, \pi_{BH})$ *with positive probability, then either* $(\pi_{AH}, \pi_{BL}, \pi_{BH}) = (0, 0, 0)$ *or* $(\pi_{AH}, \pi_{BL}, \pi_{BH}) = (0, 0, \pi^*_{BH})$, *where π^*_{BH} solves equation 11. In other words, learning is either complete or confounded.*

Proof. This follows directly from Lemma 6 and Proposition 8. ■

5 Complete Learning is Globally Stable

In the last section, we show that long run learning needs not to be complete despite the existence of uninformed players. In this section, we show that although complete learning will not arise for sure, for a generic prior it will arise with strictly positive probability. This is true even if private signal is of bounded strength. Therefore, the existence of unknown proportion of uninformed players still helps long-run learning.

To make the statement slightly precise, without loss of generality we denote posterior belief in period t_0 as Λ, and we prove that

$$\Pr(\lim_{t\to\infty} \Lambda_t = (0,0,0)|\Lambda) > 0.$$

The proof consists of two parts. In the first part (Lemma 10), we prove that: whatever current belief Λ the society holds, after observing some history $\mathfrak{h}^T_{t_0}$, the updated posterior belief $\Lambda(\mathfrak{h}^T_{t_0}|\Lambda) = (\lambda^{AH}(\mathfrak{h}^T_{t_0}|\Lambda), \lambda^{BL}(\mathfrak{h}^T_{t_0}|\Lambda), \lambda^{BH}(\mathfrak{h}^T_{t_0}|\Lambda))$ must have its third component strictly below π^*_{BH}. In the second part (equation 14), we use Fatou's lemma to prove that: a process Λ_t starting from $\Lambda(\mathfrak{h}^T_{t_0}|\Lambda)$ must converge to $(0,0,0)$ with strictly positive probability. This is driven by the facts that λ^{BH}_t is a martingale and its initial value $\lambda^{BH}(\mathfrak{h}^T_{t_0}|\Lambda)$ is strictly below the confounded value π^*_{BH}. Assume oppositely that Λ_t converges to the confounded learning point $(0,0,\pi^*_{BH})$ with probability 1. Then λ^{BH}_t must converge to π^*_{BH} for sure. But this contradicts Fatou's lemma: the expectation of the limit of λ^{BH}_t is π^*_{BH}, and is bigger than the limit of the expectation of λ^{BH}_t, which is $\lambda^{BH}(\mathfrak{h}^T_{t_0}|\Lambda)$ since λ^{BH}_t is a martingale. This violates Fatou's lemma which states that the limit of expectations must be no less than the expectation of the limit.

Recall that any finite history happens with strictly positive probability. In Lemma 10, we proves that posterior belief moves from Λ to $\Lambda(\mathfrak{h}^T_{t_0}|\Lambda)$ with positive probability. In the second part, we show that posterior belief move from $\Lambda(\mathfrak{h}^T_{t_0}|\Lambda)$ to $(0,0,0)$ with positive probability. Therefore, we can conclude that with positive probability posterior belief moves from Λ to $(0,0,0)$. In other words, complete learning must arise with strictly positive probability. Below is the formal proof.

Lemma 10 *Given any prior belief Λ_0 that allows for confounded learning, for all current belief $\Lambda \in \mathbb{R}^3_{++}$, there exists a finite sequence of actions $\mathfrak{h}^T_{t_0}$ such that*

$$\lambda^{BH}(\mathfrak{h}^T_{t_0}|\Lambda) < \pi^*_{BH},$$

where $(0, 0, \pi^*_{BH})$ is the unique confounded learning point.

Proof. Starting from any current belief Λ, if $\lambda^{BH} \geq \pi^*_{BH}$, we construct following action sequence

$$\mathfrak{h}^T_t = \begin{cases} a; \text{ if } \lambda_t \leq \pi^*_{BH}; \\ b; \text{ if } \lambda_t > \pi^*_{BH}. \end{cases}$$

Here $\lambda_t \equiv \frac{\lambda^{BH}_t + \lambda^{BL}_t}{1+\lambda^{AH}_t}$ is a random variable defined for any posterior belief Λ_t. It represents the likelihood ratio for payoff state being B over A under Λ_t.

It is directly to verify that $\frac{\phi(a|BH,\lambda_t)}{\phi(a|AL,\lambda_t)} < 1$ iff $\lambda_t < \pi^*_{BH}$; $\frac{\phi(b|BH,\lambda_t)}{\phi(b|AL,\lambda_t)} < 1$ iff $\lambda_t > \pi^*_{BH}$; and that $\frac{\phi(a|BH,\lambda_t)}{\phi(a|AL,\lambda_t)} = \frac{\phi(b|BH,\lambda_t)}{\phi(b|AL,\lambda_t)} = 1$ iff $\lambda_t = \pi^*_{BH}$. In other words, if $\lambda_t < \pi^*_{BH}$, then observing action a reduces λ^{BH}; if $\lambda_t > \pi^*_{BH}$, then observing action b reduces λ^{BH}.

Therefore, conditional on observing any action in the sequence $\mathfrak{h}^T$, λ^{BH} must decreases. If there exists infinitely many decreases which are bounded away from 0, then λ^{BH} must eventually decreases below π^*_{BH}. This is equivalent to show that: $\exists \varepsilon > 0$ and and a subsequence t_k, such that λ_{t_k} is ε away from π^*_{BH}. This is further equivalent to show that: conditional on observing $\mathfrak{h}^T$, λ_t cannot converge to π^*_{BH}. We shall show such convergence is impossible.

To show this, we need the following observation: if $\Lambda_t \in \{\Lambda_t \in \mathbb{R}^3_{++} | \lambda_t \in [\lambda_0, \pi^*_{BH}]\}$, then conditional on observing action a, λ must decrease. It is direct to verify that

$$\lambda_t \in [\lambda_0, \pi^*_{BH}] \Rightarrow \frac{\phi(a|BH, \Lambda_t)}{\phi(a|AL, \Lambda_t)} \leq 1, \frac{\phi(a|BL, \Lambda_t)}{\phi(a|AL, \Lambda_t)} < 1; \frac{\phi(a|AH, \Lambda_t)}{\phi(a|AL, \Lambda_t)} > 1. \tag{12}$$

This observation follows from that:

$$\lambda_{t+1}(a|\Lambda_t) = \frac{\lambda^{BH}_t \frac{\phi(a|BH,\Lambda_t)}{\phi(a|AL,\Lambda_t)} \downarrow + \lambda^{BL}_t \frac{\phi(a|BL,\Lambda_t)}{\phi(a|AL,\Lambda_t)} \downarrow}{1 + \lambda^{AH}_t \frac{\phi(a|AH,\Lambda_t)}{\phi(a|AL,\Lambda_t)} \uparrow} < \lambda_t, \tag{13}$$

as long as $\Lambda_t \in \mathbb{R}^3_{++}$.

This observation has the following implication: if at period $\bar{t}$, $\lambda_{\bar{t}} \leq \pi^*_{BH}$, then λ_t has to first move away from π^*_{BH}. It cannot move close to π^*_{BH} until it drops below λ_0. Therefore, if $\lambda_t \to \pi^*_{BH}$, it must eventually approach π^*_{BH} from above.

Since $\lambda_t > \pi^*_{BH}$ eventually, there exists a finite $\bar{t}$ such that $\lambda_t > \pi^*_{BH}$ for all $t > \bar{t}$. Then by construction of $\mathfrak{h}^T$, from period $\bar{t}$, only action b is observable. It is direct to that $\frac{\phi(b|BL,\Lambda_t)}{\phi(b|AL,\Lambda_t)} > 1$ always hold. So λ^{BL}_t must increase to $+\infty$. With assumption that $\lambda_t \to \pi^*_{BH}$, that $\lambda^{BL}_t \to +\infty$ implies that $\lambda^{AH}_t \to \infty$. But we can verify that: observing action b while $\lambda_t > \pi^*_{BH}$ must reduce λ^{AH}. So λ^{AH}_t is bounded above by $\lambda^{AH}_{\bar{t}}$. ■

Conditional on observing $\mathfrak{h}^T_{t_0}$, if $(\lambda^{AH}_t, \lambda^{BL}_t, \lambda^{BH}_t)$ converges to the confounded learning point $(0, 0, \pi^*_{BH})$ with probability 1, then

$$\pi^*_{BH} = E[\lim_{t\to+\infty} \lambda^{BH}_t | AL, \lambda^{BH}(\mathfrak{h}^T_{t_0}|\Lambda)] > \lambda^{BH}(\mathfrak{h}^T_{t_0}|\Lambda) = \lim_{t\to\infty} E[\lambda^{BH}_t | AL, \lambda^{BH}(\mathfrak{h}^T_{t_0}|\Lambda)]. \quad (14)$$

Here the first equation follows from the assumption that posterior belief converges to confounded learning point with probability 1; the second equation follows from the fact that λ^{BH}_t is a martingale conditional on AL and $\mathfrak{h}^T_{t_0}$. But this violates Fatou's lemma. Therefore, conditional on $\mathfrak{h}^T_{t_0}$, complete learning must arise with strictly positive probability. We also note that the probability of observing action sequence $\mathfrak{h}^T_{t_0}$ is strictly positive since this sequence is finite. So we have the following result:

Theorem 11 *In an observational learning model with unknown proportion of uninformed players, given any prior Λ_0 that admits confounded learning, for all possible current belief $\Lambda \in \mathbb{R}^3_{++}$; complete learning arise with strictly positive probability.*

6 Confounded Learning could be Globally Stable

In the previous section, we show that complete learning shall arise with strictly positive probability. In this section, we derive sufficient conditions for a similar result to hold for confounded learning.

The first result we have is that confounded learning is "locally stable": if society's current posterior belief Λ_t is sufficiently close to the confounded learning, with positive probability posterior beliefs settle down to the confounded learning. This result is obtained as a corollary of Theorem C.2 in Smith and Sørensen (2000). Below we give a rigorous statement.

A rigorous definition for a stationary point of a stochastic process to be locally stable is given as following:

Definition 12 (Locally Stable Stationary Point) *Let $(\Omega, \mathbb{P}, \mathcal{F}_t)$ be a generic filtered probability space, and $\{\Lambda_t\} : \mathbb{N} \times \Omega \to \mathbb{R}^n$ be an adapted discrete-time stochastic process. Then a stationary point $\Lambda^* \in \mathbb{R}^n$ is locally stable if there exists an open neighborhood $U \ni \Lambda^*$ such that*

$$\mathbb{P}(\{\lim_{k\to+\infty} \Lambda_{t_0+k}(\omega) = \Lambda^* | \Lambda_{t_0} \in U\}) > 0.$$

Theorem 13 *Assume there exists $(0, 0, \pi^*_{BH})$ satisfying equation 11 so that confounded learning exists. If belief updating rule $\varphi(\alpha, \lambda^{BH}_t) = \lambda^{BH}_t \frac{\phi(\alpha|BH,\Lambda_t)}{\phi(\alpha|AL,\Lambda_t)}$ weakly increases in λ^{BH}_t around $(0, 0, \pi^*_{BH})$ for $\alpha \in \{a, b\}$, then $(0, 0, \pi^*_{BH})$ is locally stable.*

Proof. See Appendix B. ■

To strengthen the local stability of confounded learning into global stability, we need to show: whatever society's current belief is, society's posterior belief moves into the local neighborhood U with positive probability. In the rest of this section, we are going to show this. For any given current belief $\Lambda \in \mathbb{R}^3_{++}$, and any $\varepsilon > 0$, we construct a finite sequence of actions $\mathfrak{h}^C_{t_0}$. Conditional on current belief Λ and observing actions sequence $\mathfrak{h}^C_{t_0}$, society's posterior belief moves into the pre-determined ε-neighborhood of confounded learning Λ^*. Since any finite sequence of actions happens with strictly positive probability, we can obtain the global stability of confounded learning from the existence of $\mathfrak{h}^C_{t_0}$.

The $\mathfrak{h}^C_{t_0}$ is constructed in two phases. We first construct an infinite action sequence $\mathfrak{h}^{C_1}$ that can push society's belief arbitrarily close to axis λ^{BH}. In other words, in the end of the first phase, society's posterior belief Λ must satisfy that λ^{AH} and λ^{BL} are sufficiently close to 0. By doing so, we roughly turn the global stability problem from a three-dimension problem into a one-dimension problem. Then, in the second phase, we construct an action sequence consists of action b to push society's belief into the pre-determined $\varepsilon-$neighborhood along the direction of axis-λ^{BH}.

Intuitively, construction in phase I is done in the following way [5] : given any current belief $\Lambda_t \in \mathbb{R}^3_{++}$, select the action that reduces $\frac{\lambda^{AH}_t}{\lambda^{BH}_t} + \frac{\lambda^{BL}_t}{\lambda^{BH}_t}$. For a generic Λ_t, we can always reduce $\frac{\lambda^{AH}_t}{\lambda^{BH}_t} + \frac{\lambda^{BL}_t}{\lambda^{BH}_t}$ for that

$$(\frac{\lambda^{AH}_t(a)}{\lambda^{BH}_t(a)}, \frac{\lambda^{BL}_t(a)}{\lambda^{BH}_t(a)}) - (\frac{\lambda^{AH}_t}{\lambda^{BH}_t}, \frac{\lambda^{BL}_t}{\lambda^{BH}_t}) = -\frac{\phi(b|BH,\Lambda_t)}{\phi(a|BH,\Lambda_t)}\Big((\frac{\lambda^{AH}_t(b)}{\lambda^{BH}_t(b)}, \frac{\lambda^{BL}_t(b)}{\lambda^{BH}_t(b)}) - (\frac{\lambda^{AH}_t}{\lambda^{BH}_t}, \frac{\lambda^{BL}_t}{\lambda^{BH}_t})\Big).$$

[5]See appendix C, especially lemma 16, for a rigorous version.

By doing so, $\frac{\lambda_t^{AH}}{\lambda_t^{BH}} + \frac{\lambda_t^{BL}}{\lambda_t^{BH}}$ form a decreasing sequence and are bounded from below, and hence must converge. We conjecture that for a generic set of learning primitives, $\frac{\lambda_t^{AH}}{\lambda_t^{BH}} + \frac{\lambda_t^{BL}}{\lambda_t^{BH}} \to 0$. Let us rewrite society's belief $P_t = (p_t^{AH}, p_t^{BL}, p_t^{BH})$ in probabilities instead of ratios. It is direct to see that $\frac{\lambda_t^{AH}}{\lambda_t^{BH}} + \frac{\lambda_t^{BL}}{\lambda_t^{BH}} = \frac{p_t^{AH}}{p_t^{BH}} + \frac{p_t^{BL}}{p_t^{BH}}$. Now, let us assume that $\frac{p_t^{AH}}{p_t^{BH}} + \frac{p_t^{BL}}{p_t^{BH}} \to c > 0$, then P_t must converges to a limit set $P_{cluster}$ which lives on the plane determined by $\frac{p^{AH}}{p^{BH}} + \frac{p^{BL}}{p^{BH}} = c$. We conjecture such a limit set $P_{cluster}$ cannot exist for a generic set of learning primitives. To see the intuition of this conjecture, let us assume that $P_{cluster} = \{P_{s_1}, P_{s_2}\}$, then we must have coordinates of $P_{cluster}$ satisfying the following equations system:

$$
\begin{aligned}
&P_{s_1}(\alpha_1) = P_{s_2};\\
&P_{s_2}(\alpha_2) = P_{s_1};\\
&\frac{p_{s_1}^{AH}}{p_{s_1}^{BH}} + \frac{p_{s_1}^{BL}}{p_{s_1}^{BH}} = c, \frac{p_{s_2}^{AH}}{p_{s_2}^{BH}} + \frac{p_{s_2}^{BL}}{p_{s_2}^{BH}} = c.
\end{aligned}
$$

Here the first row represents three equations that there must exist an action α_1 such that society's belief moves from p_{s_1} to p_{s_2} conditional on seeing α_1; the second row represents another three equations that there must exist an action α_2 such that society's belief moves from p_{s_2} to p_{s_1} conditional on seeing α_2; the two equations in the third row follows the assumption that $\frac{p_t^{AH}}{p_t^{BH}} + \frac{p_t^{BL}}{p_t^{BH}} \to c$. Therefore, if the cardinality of $P_{cluster}$ is 2, then the six coordinates in $P_{cluster}$ must solve eight equations. This seems to be impossible under a generic set of learning primitives. This intuition works if $\|P_{cluster}\| \geq 2$. In fact, the cardinality of $P_{cluster}$ cannot be 1 with assumption that $\frac{p^{AH}}{p^{BH}} + \frac{p^{BL}}{p^{BH}} = c$. [6] To move from the above intuitive conjecture to a rigorous statement, we need condition 1 in theorem 14. In other words, if condition 1 is satisfied, then $\frac{p_t^{AH}}{p_t^{BH}} + \frac{p_t^{BL}}{p_t^{BH}} \to 0$ must hold. Interest readers can refer to lemma 16 in appendix C for a detailed proof. From intuition described above and numerical experiments we performed, we believe that condition 1 holds for a generic set of learning primitives.

The ultimate goal of construction in phase I is to push society's belief sufficiently close to axis λ^{BH}, which is a stronger statement than $\frac{\lambda_t^{AH}}{\lambda_t^{BH}} + \frac{\lambda_t^{BL}}{\lambda_t^{BH}} \to 0$. After all, the ratio goes to 0 could happen if $\lambda_t^{AH}, \lambda_t^{BL}$ are large, but λ_t^{BH} increases fast enough. If this is the case, $\lambda_t = \frac{\lambda_t^{BH} + \lambda_t^{BL}}{1 + \lambda_t^{AH}} \to +\infty$. We can actually compute the long run frequency of each action if $\lambda_t \to +\infty$ in a sub-sequence t_k. (See lemma 23 in appendix C for a detailed computation.)

[6] If $\|P_{cluster}\| = 1$, then the posterior belief in ratios Λ_s corresponding to $p_s \in P_{cluster}$ must satisfy $\Lambda_s \in \{0, +\infty\} \times \{0, +\infty\} \times \{0, \pi_{BH}^*, +\infty\}$. We could verify that no such Λ_s can be stationary and satisfy $\frac{p_s^{AH}}{p_s^{BH}} + \frac{p_s^{BL}}{p_s^{BH}} = c > 0$.

Such long run frequencies imply that $\lambda_{t_k}^{AH} \to 0$ and $\lambda_{t_k}^{BH} \to 0$ if and only if condition 2 in theorem 14 holds.

Therefore, with condition 1 and 2, we can push society's belief arbitrarily close to axis λ^{BH}. Depending on the ε in the pre-determined $\varepsilon-$neighborhood, we can determine a proper period to stop pushing the belief closer. And the construction in phase I is complete.

Let us denote the society's belief at the end of phase I as Λ_I. As long as $\lambda^{AH}, \lambda^{BL}$ are negligible comparing to λ^{BH}, to push the belief towards Λ^*, we just need to push λ^{BH} towards π_{BH}^*. This can be done by action b for that $\frac{\phi(b|BH,\lambda)}{\phi(b|AL,\lambda)} < 1$ if $\lambda > \pi_{BH}$ and that $\frac{\phi(b|BH,\lambda)}{\phi(b|AL,\lambda)} > 1$ if $\lambda < \pi_{BH}$. [7] With condition 4 in theorem 14, λ^{BH} can not jump across π_{BH}^*. Therefore, we could use a long sequence of action b to push society's belief from Λ_I into the pre-determined $\varepsilon-$neighborhood, provided that $\frac{\lambda^{AH}}{\lambda^{BH}} + \frac{\lambda^{BL}}{\lambda^{BH}}$ stays close to 0.

The only thing needs to worry in phase II is that $\frac{\lambda^{BL}}{\lambda^{BH}}$ may increases too much, which implies that λ^{BL} is no longer negligible, comparing to λ^{BH}. [8] In general, we can control the ratio of $\frac{\lambda^{BL}}{\lambda^{BH}}$ in phase II by shrinking it really small in phase I. However, shrinking $\frac{\lambda^{BL}}{\lambda^{BH}}$ doesn't solve the problem if $\lambda_t^{BH}(\mathfrak{h}_t^{C_1}|\Lambda) \to +\infty$. If this is the case, then shrinking $\frac{\lambda^{BL}}{\lambda^{BH}}$ in phase I comes at the cost of λ^{BH} explodes, and a super long sequence of actions b to push λ^{BH} close to π_{BH}^* in phase II. It is not clear that $\frac{\lambda^{BL}}{\lambda^{BH}}$ stays negligible after seeing a super long sequence of actions b, even if it starts with a super small value. In proposition 31 we deal with this situation. With condition 3 in theorem 14, we can always push the society's belief into a position where λ^{BH} is bounded above while $\frac{\lambda^{BL}}{\lambda^{BH}}$ is arbitrarily small. In Figure 1, an example of beliefs' movement in phase II is depicted.

The set of learning primitives that satisfy condition 2 and 3 in theorem 14 are open. Furthermore, from numerical examples, we conjecture that condition 3 actually holds for all learning primitives. Therefore, we believe that global stability of confounded learning is a robust phenomenon which arises under sufficiently many learning environments.

To summarize, we have the following theorem:

Theorem 14 *If prior $\Lambda_0 \in PB$, then for any current belief $\Lambda_t \in \mathbb{R}_{++}^3$ and $\varepsilon > 0$. If*

1. $\mathfrak{F}(x)\frac{\phi(b|AH,x)}{\phi(b|BL,x)} < \mathfrak{F}(y)$, *where* $\mathfrak{F}(x) = \frac{\phi(b|BL,x)-\phi(b|BH,x)}{\phi(b|BH,x)-\phi(b|AH,x)}$ *on* $x \in [x_{BH}, 1]$ *and* $y = \frac{x\phi(b|BH,x)}{(1-x)\phi(b|AL,x)+x\phi(b|BH,x)}$;
2. $\frac{\log\phi(a|AH,1)-\log\phi(a|BL,1)}{\log\phi(b|BL,1)-\log\phi(b|AH,1)} > \frac{\log\phi(a|AH,1)-\log\phi(a|AL,1)}{\log\phi(b|AL,1)-\log\phi(b|AH,1)}$;

[7] $\lambda = \frac{\lambda^{BH}+\lambda^{BL}}{1+\lambda^{AH}} \approx \lambda^{BH}$ if $\frac{\lambda^{AH}}{\lambda^{BH}} + \frac{\lambda^{BL}}{\lambda^{BH}}$ is sufficiently small.

[8] We don't need to worry about λ^{AH} since $\frac{\lambda^{AH}}{\lambda^{BH}}$ always decreases conditional on observing action b. Therefore, as long as λ^{AH} is negligible to λ^{BH} in the beginning of phase II, it must stay negligible.

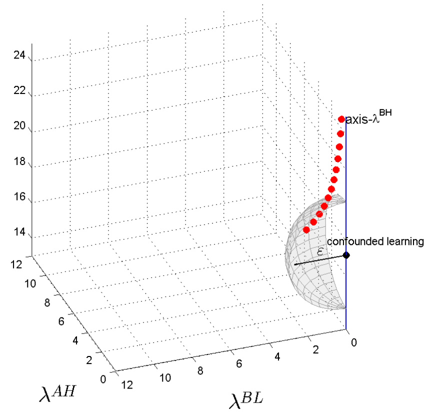

Figure 1: Belief movements in phase 2

3. $\frac{\log\phi(a|AL,1)-\log\phi(a|BL,1)}{\log\phi(b|BL,1)-\log\phi(b|AL,x)} > \frac{\log\phi(a|BH,1)-\log\phi(a|AL,1)}{\log\phi(b|AL,1)-\log\phi(b|BH,1)}$;

4. $\lambda^{BH}\frac{\phi(b|BH,\frac{\lambda^{BH}}{\lambda^{BH}+1})}{\phi(b|AL,\frac{\lambda^{BH}}{\lambda^{BH}+1})}$ *strictly increases in* λ^{BH}.

then there exists a finite sequence of actions $\mathfrak{h}^C_{t_0}$*, such that*

$$\|\Lambda_{t+t_0}(\mathfrak{h}^C_{t_0}|\Lambda_t) - \Lambda^*\| < \varepsilon.$$

In other words, starting from Λ_t*, after seeing* $\mathfrak{h}^C_{t_0}$*, the society's posterior belief enters the* $\varepsilon-$*neighborhood of confounded learning.*

Furthermore, by local stability of confounded learning Λ^**,* $\exists\ \varepsilon_0 > 0$*, such that*

$$\|\Lambda_{t+t_0}(\mathfrak{h}^C_{t_0}|\Lambda_t) - \Lambda^*\| < \varepsilon_0 \Rightarrow \lim_{k\to+\infty}\Lambda_{t+t_0+k} = \Lambda^* \text{ with positive probability.}$$

So Λ^* *is globally stable under above conditions.*

Proof. See Appendix C. ■

7 Conclusion

We study the effect of uninformed players on long run learning in an observational learning model. Because uninformed players act exclusively on their own signals, their actions keep generating new information. We argue that if the proportion of uninformed players is unknown and rational players need to simultaneously learn the true proportion and the payoff-relevant state, then confounded learning could arise. We further show that complete learning is globally stable: for a large set of priors, starting from any current belief, society's belief settles down to complete learning with positive probability. We also give sufficient conditions that guarantee confounded learning to be globally stable.

A Proof of Lemma 4

We first compute the evolution rule of $\lambda_t^{\omega_1\omega_2}$. Conditional on seeing action $\alpha \in \{a, b\}$, we have

$$\lambda_{t+1}^{\omega_1\omega_2}(h_t,\alpha) \doteq \frac{\mathbb{P}_t(\omega_1\omega_2|h_t,\alpha)}{\mathbb{P}_t(AL|h_t,\alpha)} = \frac{\mathbb{P}_{t-1}(\omega_1\omega_2|h_t)}{\mathbb{P}_{t-1}(AL|h_t)}\frac{\phi(\alpha|\omega_1\omega_2,\Lambda_t(h_t))}{\phi(\alpha|AL,\Lambda_t(h_t))} = \lambda_t^{\omega_1\omega_2}(h_t)\frac{\phi(\alpha|\omega_1\omega_2,\Lambda_t(h_t))}{\phi(\alpha|AL,\Lambda_t(h_t))}; \quad (15)$$

Using evolution rule 15, we have

$$\begin{aligned}
& E[\lambda_{t+1}^{\omega_1\omega_2}|AL,h_t] \\
= \ & \lambda_{t+1}^{\omega_1\omega_2}(h_t,a)\phi(a|AL,\Lambda_t(h_t)) + \lambda_{t+1}^{\omega_1\omega_2}(h_t,b)\phi(b|AL,\Lambda_t(h_t)) \\
= \ & [\lambda_t^{\omega_1\omega_2}(h_t)\frac{\phi(a|\omega_1\omega_2,\Lambda_t(h_t))}{\phi(a|AL,\Lambda_t(h_t))}]\phi(a|AL,\Lambda_t(h_t)) + [\lambda_t^{\omega_1\omega_2}(h_t)\frac{\phi(b|\omega_1\omega_2,\Lambda_t(h_t))}{\phi(b|AL,\Lambda_t(h_t))}]\phi(b|AL,\Lambda_t(h_t)) \\
= \ & \lambda_t^{\omega_1\omega_2}(h_t). \qquad (16)
\end{aligned}$$

It is obvious that $\lambda_t^{\omega_1\omega_2}$ is non-negative since it is a likelihood ratio. This completes the proof.

B Proof of Theorem 13

For reader's convenience, we first rewrite Theorem C.2 of Smith and Sørensen (2000) in our notations.

Theorem 15 *Let $\langle(\alpha_t,\Lambda_t)\rangle$ be a discrete-time Markov Process on $\{a,b\}\times\mathbb{R}^3$, with transitions*

$$\Lambda_{t+1} = \varphi(\alpha_t,\Lambda_t), \textit{with prob } \phi(\alpha_t|AL,\Lambda_t).$$

Let Λ^* be a fixed point of $\varphi(\alpha,\cdot)$. If

1. *$\phi(\alpha|AL,\Lambda^*)$ is continuous at Λ^*, and $\varphi(\alpha,\cdot)$ is $\mathcal{C}^1$ at Λ^*;*

2. *$D_\alpha\varphi(\alpha,\Lambda^*)$ has distinct, real, positive, non-unit eigenvalue;*

3. $\phi(a|AL,\Lambda^*)D_a\varphi(a,\Lambda^*)+\phi(b|AL,\Lambda^*)D_b\varphi(b,\Lambda^*)=I$*.*

Then, Λ^* is locally stable.

It is straightforward to verify that $\phi(\alpha|AL,\Lambda^*)$ is continuous and that $\varphi(\alpha,\cdot)$ is $\mathcal{C}^1$ at Λ^*. We further compute

$$D_a\varphi(a,\Lambda^*)=\begin{bmatrix}\frac{\phi(a|AH,\Lambda^*)}{\phi(a|AL,\Lambda^*)} & 0 & 0\\ 0 & \frac{\phi(a|BL,\Lambda^*)}{\phi(a|AL,\Lambda^*)} & 0\\ -(\frac{\pi_{BH}}{\pi_{BH}+1})^2G_1 & \frac{\pi_{BH}}{(\pi_{BH}+1)^2}G_1 & 1+\frac{\pi_{BH}}{(\pi_{BH}+1)^2}G_1\end{bmatrix}$$

$$D_b\varphi(b,\Lambda^*)=\begin{bmatrix}\frac{\phi(b|AH,\Lambda^*)}{\phi(b|AL,\Lambda^*)} & 0 & 0\\ 0 & \frac{\phi(b|BL,\Lambda^*)}{\phi(b|AL,\Lambda^*)} & 0\\ -(\frac{\pi_{BH}}{\pi_{BH}+1})^2G_2 & \frac{\pi_{BH}}{(\pi_{BH}+1)^2}G_2 & 1+\frac{\pi_{BH}}{(\pi_{BH}+1)^2}G_2\end{bmatrix}$$

where

$$G_1=\frac{(1-p_L)f^A(\frac{\pi_{BH}}{\pi_{BH}+1})-(1-p_H)f^B(\frac{\pi_{BH}}{\pi_{BH}+1})}{\phi(a|AL,\Lambda^*)},$$

and

$$G_2=\frac{-(1-p_L)f^A(\frac{\pi_{BH}}{\pi_{BH}+1})+(1-p_H)f^B(\frac{\pi_{BH}}{\pi_{BH}+1})}{\phi(b|AL,\Lambda^*)}.$$

Then it is straightforward to verify that $\phi(a|AL,\Lambda^*)D_a\varphi(a,\Lambda^*)+\phi(b|AL,\Lambda^*)D_b\varphi(b,\Lambda^*)=I$ holds. Furthermore, let

$$Q=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ \frac{\left(\frac{\pi_{BH}}{\pi_{BH}+1}\right)^2G_1}{1+\frac{\pi_{BH}}{(\pi_{BH}+1)^2}G_1-\frac{\phi(a|AH,\pi_{BH})}{\phi(a|AL,\pi_{BH})}} & \frac{-\frac{\pi_{BH}}{(\pi_{BH}+1)^2}G_1}{1+\frac{\pi_{BH}}{(\pi_{BH}+1)^2}G_1-\frac{\phi(a|BL,\pi_{BH})}{\phi(a|AL,\pi_{BH})}} & 1\end{bmatrix}$$

Then we can verify that $Q^{-1}D_\alpha\varphi(\alpha,\cdot)Q=M_\alpha$, where

$$M_a=\begin{bmatrix}\frac{\phi(a|AH,\Lambda^*)}{\phi(a|AL,\Lambda^*)} & 0 & 0\\ 0 & \frac{\phi(a|BL,\Lambda^*)}{\phi(a|AL,\Lambda^*)} & 0\\ 0 & 0 & 1+\frac{\pi_{BH}}{(\pi_{BH}+1)^2}G_1\end{bmatrix}$$

$$M_b=\begin{bmatrix}\frac{\phi(b|AH,\Lambda^*)}{\phi(b|AL,\Lambda^*)} & 0 & 0\\ 0 & \frac{\phi(b|BL,\Lambda^*)}{\phi(b|AL,\Lambda^*)} & 0\\ 0 & 0 & 1+\frac{\pi_{BH}}{(\pi_{BH}+1)^2}G_2.\end{bmatrix}$$

We observe that $G_1 > 0$ and $G_2 < 0$ since $\pi_{BH} > \frac{1-p_H}{1-p_L}$ as in proposition 8 and that $\frac{f^B(x)}{f^A(x)} = \frac{1-x}{x}$. Then it is straightforward that $D_\alpha \varphi(\alpha, \Lambda^*)$, $\alpha \in \{a, b\}$ have real, distinct and non-unit eigenvalues. Finally, with assumption that $\frac{\partial \varphi_3(a_t, \Lambda^*)}{\lambda^{BH}} > 0$, we have $1 + \frac{\pi_{BH}}{(\pi_{BH}+1)^2} G_2 > 0$. So all the eigenvalues are positive as well.

C Omitted Proofs in Global Stability

In this section, we first explicitly construct an action sequence $\mathfrak{h}^{C_1}$. In lemmas 16 and 24, we prove that society's posterior belief can be arbitrarily close to axis λ^{BH} conditional on seeing sufficiently many actions in $\mathfrak{h}^{C_1}$. In lemmas 27 and 28, we prove that society's belief, starting from a position sufficiently close to axis λ^{BH} and is bounded above by a finite number $\overline{\lambda}^{BH} < \frac{\bar{s}}{1-\bar{s}}$ [9] , can eventually enter any pre-determined $\varepsilon-$neighborhood of confounded learning after observing a long sequence of action b. In proposition 31, we show that we can always push society's belief into a position sufficiently close to axis$-\lambda^{BH}$ and is bounded above by a proper $\overline{\lambda}^{BH}$. A lot of computation results are used in the proofs. To not to disrupt the logic of proofs, we verify these computation results in the end of this section, from claim 32 to claim 37.

We arbitrarily choose and fix a current belief $\Lambda \in \mathbb{R}^3_{++}$ and a $\varepsilon > 0$ in this section. We use $\Lambda(h|\Lambda_1)$ to represent the posterior belief updated from Λ_1 after seeing history h.

At period t, action $\mathfrak{h}^{C_1}_t$ is chosen to reduce the ratio $\frac{\lambda^{AH}_t}{\lambda^{BH}_t} + \frac{\lambda^{BL}_t}{\lambda^{BH}_t}$. We observe that

$$\begin{aligned} &\left(\frac{\lambda^{AH}_t}{\lambda^{BH}_t}\left(\frac{\phi(a|AH,\Lambda_t)}{\phi(a|BH,\Lambda_t)} - 1\right), \frac{\lambda^{BL}_t}{\lambda^{BH}_t}\left(\frac{\phi(a|BL,\Lambda_t)}{\phi(a|BH,\Lambda_t)} - 1\right)\right) \\ = \; & -\frac{\phi(b|BH,\Lambda_t)}{\phi(a|BH,\Lambda_t)}\left(\frac{\lambda^{AH}_t}{\lambda^{BH}_t}\left(\frac{\phi(b|AH,\Lambda_t)}{\phi(b|BH,\Lambda_t)} - 1\right), \frac{\lambda^{BL}_t}{\lambda^{BH}_t}\left(\frac{\phi(b|BL,\Lambda_t)}{\phi(b|BH,\Lambda_t)} - 1\right)\right). \end{aligned} \tag{17}$$

Therefore, if we consider the pair $\left(\frac{\lambda^{AH}_t}{\lambda^{BH}_t}, \frac{\lambda^{BL}_t}{\lambda^{BH}_t}\right)$, after seeing an action, it can only moves toward two opposite directions. Therefore, generically we can choose an action to reduce $\frac{\lambda^{AH}_t}{\lambda^{BH}_t} + \frac{\lambda^{BL}_t}{\lambda^{BH}_t}$.

Following this observation, $\mathfrak{h}^{C_1}$ is constructed in the following way: at period t, if there exists an action $\alpha \in \{a, b\}$ such that $\frac{\lambda^{AH}_{t+1}(\alpha)}{\lambda^{BH}_{t+1}(\alpha)} + \frac{\lambda^{BL}_{t+1}(\alpha)}{\lambda^{BH}_{t+1}(\alpha)} < \frac{\lambda^{AH}_t}{\lambda^{BH}_t} + \frac{\lambda^{BL}_t}{\lambda^{BH}_t}$, then $\mathfrak{h}^{C_1}_t = \alpha$; otherwise, choose action a. From the construction, $\frac{\lambda^{AH}_t(\mathfrak{h}^{C_1}_t|\Lambda)}{\lambda^{BH}_t(\mathfrak{h}^{C_1}_t|\Lambda)} + \frac{\lambda^{BL}_t(\mathfrak{h}^{C_1}_t|\Lambda)}{\lambda^{BH}_t(\mathfrak{h}^{C_1}_t|\Lambda)}$ obviously form a decreasing

[9]In this section, most of the times, we don't explicitly distinguish bounded private signal and unbounded private signal. If private signal is unbounded, we understand that $\frac{1-\bar{s}}{\bar{s}} \equiv= +\infty$

sequence bounded from below by 0. The following lemma shows that it must converge to 0 with condition 18.

Lemma 16 *Let* $x_{BH} = \frac{\pi^*_{BH}}{\pi^*_{BH}+1}$. *For all* $x \in [x_{BH}, 1]$, *let* $\mathfrak{F}(x) = \frac{\phi(b|BL,x)-\phi(b|BH,x)}{\phi(b|BH,x)-\phi(b|AH,x)}$, *if*

$$\mathfrak{F}(x)\frac{\phi(b|AH,x)}{\phi(b|BL,x)} < \mathfrak{F}(y), \text{ where } y = \frac{x\phi(b|BH,x)}{(1-x)\phi(b|AL,x)+x\phi(b|BH,x)} \tag{18}$$

then there exists an infinite sequence $\mathfrak{h}^{C_1}$ *such that*

$$\lim_{t\to+\infty}\frac{\lambda_t^{AH}(\mathfrak{h}_t^{C_1}|\Lambda)}{\lambda_t^{BH}(\mathfrak{h}_t^{C_1}|\Lambda)}+\frac{\lambda_t^{BL}(\mathfrak{h}_t^{C_1}|\Lambda)}{\lambda_t^{BH}(\mathfrak{h}_t^{C_1}|\Lambda)}=0,$$

where Λ *is the arbitrarily chosen current belief at the beginning of this section.*

For notation convenience, from now on in the proof of lemma 16, we drop $\mathfrak{h}^{C_1}$ with the understanding that Λ_t is actually $\Lambda(\mathfrak{h}_t^{C_1}|\Lambda)$. For example, when we write λ_t^{AH}, we mean a number $\lambda_t^{AH}(\mathfrak{h}_t^{C_1}|\Lambda)$, rather than a random variable.

Proof of lemma 16. Since $\frac{\lambda_t^{AH}}{\lambda_t^{BH}}+\frac{\lambda_t^{BL}}{\lambda_t^{BH}}$ form a decreasing sequence bounded from below, it converges for sure. Let's assume it converges to a positive constant c. Following sequence of claims lead to a contradiction.

Recall that $x_t = \frac{\lambda_t^{BH}+\lambda_t^{BL}}{1+\lambda_t^{AH}+\lambda_t^{BL}+\lambda_t^{BH}}$, following two claims 17 and 18 says that eventually x_t must stay strictly above x_0.

Claim 17 $\nexists$ *infinite sub-sequence* t_k *such that* $x_{t_k} \to x_0$.

Proof. Assume the opposite. By the construction, we have $\frac{\lambda_t^{AH}}{\lambda_t^{BH}}+\frac{\lambda_t^{BL}}{\lambda_t^{BH}}$ monotonically decreases and is bounded from below, so

$$\begin{aligned}
&\lim_{t_k\to+\infty}[\frac{\lambda_{t_k+1}^{AH}}{\lambda_{t_k+1}^{BH}}+\frac{\lambda_{t_k+1}^{BL}}{\lambda_{t_k+1}^{BH}}]-[\frac{\lambda_{t_k}^{AH}}{\lambda_{t_k}^{BH}}+\frac{\lambda_{t_k}^{BL}}{\lambda_{t_k}^{BH}}]\\
=&\lim_{t_k\to+\infty}[\frac{\lambda_{t_k}^{AH}}{\lambda_{t_k}^{BH}}\frac{\phi(\alpha_{t_k}|AH,x_{t_k})-\phi(\alpha_{t_k}|BH,x_{t_k})}{\phi(\alpha_{t_k}|BH,x_{t_k})}+\frac{\lambda_{t_k}^{BL}}{\lambda_{t_k}^{BH}}\frac{\phi(\alpha_{t_k}|BL,x_{t_k})-\phi(\alpha_{t_k}|BH,x_{t_k})}{\phi(\alpha_{t_k}|BH,x_{t_k})}]\\
=&\ 0.
\end{aligned}$$

Fact 34 (verified in the end of this section) says that $\frac{\phi(\alpha_{t_k}|AH,x_{t_k})-\phi(\alpha_{t_k}|BH,x_{t_k})}{\phi(\alpha_{t_k}|BH,x_{t_k})}$ is strictly bounded away from 0. The assumption that $x_{t_k}\to x_0$ implies that $\frac{\phi(\alpha_{t_k}|BL,x_{t_k})-\phi(\alpha_{t_k}|BH,x_{t_k})}{\phi(\alpha_{t_k}|BH,x_{t_k})}\to 0$. Therefore, we must have $\frac{\lambda_{t_k}^{AH}}{\lambda_{t_k}^{BH}}\to 0$. Furthermore, since we assume $\lim_{t\to+\infty}\frac{\lambda_t^{AH}}{\lambda_t^{BH}}+\frac{\lambda_t^{BL}}{\lambda_t^{BH}}=c$, we must also have $\frac{\lambda_{t_k}^{BL}}{\lambda_{t_k}^{BH}} \to c$.

To summarize, with assumption that

$$\frac{\lambda_t^{AH}}{\lambda_t^{BH}} + \frac{\lambda_t^{BL}}{\lambda_t^{BH}} \to c; \text{ and } \exists t_k \, s.t. \, x_{t_k} \to x_0; \tag{19}$$

we must have

$$\frac{\lambda_{t_k}^{AH}}{\lambda_{t_k}^{BH}} \to 0; \frac{\lambda_{t_k}^{BL}}{\lambda_{t_k}^{BH}} \to c; x_{t_k} \to x_0 \tag{20}$$

This implies the limit position of Λ_{t_k} must be

$$\lambda_{t_k}^{AH} \to 0; \lambda_{t_k}^{BL} \to \frac{cx_0}{(1+c)(1-x_0)}; \lambda_{t_k}^{BH} \to \frac{x_0}{(1+c)(1-x_0)}. \tag{21}$$

Now we prove that Λ_{t_k} cannot converge to above limit. For a sufficiently large t_k, let us consider the action $\mathfrak{h}_{t_k}^{C_1}$ at period t_k. If $\mathfrak{h}_{t_k}^{C_1} = a$, then x_{t_k+1} must be sufficiently close to $\frac{x_0[1-F^B(x_0)]}{x_0[1-F^B(x_0)]+(1-x_0)[1-F^A(x_0)]} \equiv x_{t_k+1}^a < x_0$. Then $\mathfrak{h}_{t_{k+1}}^{C_1}$ must be b since $\frac{\phi(b|BL,x)}{\phi(b|BH,x)} < 1$ when $x < x_0$. However, we must have $\big[\frac{\lambda_{t_k+2}^{AH}}{\lambda_{t_k+2}^{BH}} + \frac{\lambda_{t_k+2}^{BL}}{\lambda_{t_k+2}^{BH}}\big] - \big[\frac{\lambda_{t_k+1}^{AH}}{\lambda_{t_k+1}^{BH}} + \frac{\lambda_{t_k+1}^{BL}}{\lambda_{t_k+1}^{BH}}\big]$ be sufficiently close to $c\frac{\phi(a|BL,x_0)}{\phi(a|BH,x_0)}\frac{\phi(b|BL,x_{t_k+1}^a)-\phi(b|BH,x_{t_k+1}^a)}{\phi(b|BH,x_{t_k+1}^a)}$ which is strictly bounded below from 0. This contradicts that $\frac{\lambda_t^{AH}}{\lambda_t^{BH}} + \frac{\lambda_t^{BL}}{\lambda_t^{BH}}$ must converge.

Similarly, if $\mathfrak{h}_{t_k}^{C_1} = b$, then x_{t_k+1} must be sufficiently close to $\frac{x_0F^B(x_0)}{x_0F^B(x_0)+(1-x_0)F^A(x_0)} \equiv x_{t_k+1}^b > x_0$. Then $\mathfrak{h}_{t_{k+1}}^{C_1}$ must be a since $\frac{\phi(a|BL,x)}{\phi(a|BH,x)} < 1$ when $x > x_0$. We also have $\big[\frac{\lambda_{t_k+2}^{AH}}{\lambda_{t_k+2}^{BH}} + \frac{\lambda_{t_k+2}^{BL}}{\lambda_{t_k+2}^{BH}}\big] - \big[\frac{\lambda_{t_k+1}^{AH}}{\lambda_{t_k+1}^{BH}} + \frac{\lambda_{t_k+1}^{BL}}{\lambda_{t_k+1}^{BH}}\big]$ must be sufficiently close to $c\frac{\phi(b|BL,x_0)}{\phi(b|BH,x_0)}\frac{\phi(a|BL,x_{t_k+1}^b)-\phi(a|BH,x_{t_k+1}^b)}{\phi(a|BH,x_{t_k+1}^b)}$

which is also strictly bounded below 0. ■

Claim 18 $\nexists$ *infinite sub-sequence* t_k *such that* $x_{t_k} < x_0$.

Proof. Assume the opposite.

It is direct to verify that $\frac{\phi(b|AH,x)}{\phi(b|BH,x)} < 1$ and $\frac{\phi(b|BL,x)}{\phi(b|BH,x)} < 1$ if $x < x_0$. So

$$\begin{aligned}
0 &= \lim_{t_k \to +\infty} [\frac{\lambda_{t_k+1}^{AH}}{\lambda_{t_k+1}^{BH}} + \frac{\lambda_{t_k+1}^{BL}}{\lambda_{t_k+1}^{BH}}] - [\frac{\lambda_{t_k}^{AH}}{\lambda_{t_k}^{BH}} + \frac{\lambda_{t_k}^{BL}}{\lambda_{t_k}^{BH}}] \\
&= \lim_{t_k \to +\infty} [\frac{\lambda_{t_k}^{AH}}{\lambda_{t_k}^{BH}} \frac{\phi(b|AH, x_{t_k}) - \phi(b|BH, x_{t_k})}{\phi(b|BH, x_{t_k})} + \frac{\lambda_{t_k}^{BL}}{\lambda_{t_k}^{BH}} \frac{\phi(b|BL, x_{t_k}) - \phi(b|BH, x_{t_k})}{\phi(b|BH, x_{t_k})}] \\
&\leq \lim_{t_k \to +\infty} [\frac{\lambda_{t_k}^{AH}}{\lambda_{t_k}^{BH}} \frac{\phi(b|AH, x_{t_k}) - \phi(b|BH, x_{t_k})}{\phi(b|BH, x_{t_k})}] \leq 0.
\end{aligned}$$

Use fact 34, we have again

$$\frac{\lambda_{t_k}^{AH}}{\lambda_{t_k}^{BH}} \to 0; \frac{\lambda_{t_k}^{BL}}{\lambda_{t_k}^{BH}} \to c.$$

Then

$$\begin{aligned}
&\lim_{t_k \to +\infty} [\frac{\lambda_{t_k}^{AH}}{\lambda_{t_k}^{BH}} \frac{\phi(b|AH, x_{t_k}) - \phi(b|BH, x_{t_k})}{\phi(b|BH, x_{t_k})} + \frac{\lambda_{t_k}^{BL}}{\lambda_{t_k}^{BH}} \frac{\phi(b|BL, x_{t_k}) - \phi(b|BH, x_{t_k})}{\phi(b|BH, x_{t_k})}] \\
= &\lim_{t_k \to +\infty} [\frac{\lambda_{t_k}^{BL}}{\lambda_{t_k}^{BH}} \frac{\phi(b|BL, x_{t_k}) - \phi(b|BH, x_{t_k})}{\phi(b|BH, x_{t_k})}] = 0
\end{aligned}$$

implies that $x_{t_k} \to x_0$. Then we can just cite Claim 17. ■

There is a one-to-one map between Λ_t and $P_t = (p_t^{AH}, p_t^{BL}, p_t^{BH})$, which is the society's posterior belief represented by probabilities rather than ratios. We can verify that $\frac{\lambda_t^{AH}}{\lambda_t^{BH}} + \frac{\lambda_t^{BL}}{\lambda_t^{BH}} = \frac{p_t^{AH}}{p_t^{BH}} + \frac{p_t^{BL}}{p_t^{BH}}, \forall \Lambda_t \in \mathbb{R}_{++}^3$.

The following claim describes the limit of P_t under the assumption that $\frac{\lambda_t^{AH}}{\lambda_t^{BH}} + \frac{\lambda_t^{BL}}{\lambda_t^{BH}} \to c$. The limit of P_t must converges to a set $P_{cluster}$. Starting from each limit point P_s, there exists one action α. Upon seeing this action α, society's belief update from P_s to another limit belief in $P_{cluster}$.

Claim 19 $\exists P_{cluster} = \{P_s\}_{s \in I}$ *satisfying*

1. *Each $P_s \in P_{cluster}$ is a cluster point. In other words, $\exists$ sub-sequence t_k^s such that $\lim_{t_k^s \to \infty} P_{t_k^s} = P_s$.*
2. *For each $P_s = (p_s^{AH}, p_s^{BL}, p_s^{BH})$, we have $\frac{p_s^{AH}}{p_s^{BH}} + \frac{p_s^{BL}}{p_s^{BH}} = c$, and $x_s > x_0$.*
3. *For each P_s, $\exists$ at least one action $\alpha \in \{a, b\}$ such that $P_s(\alpha) \in P_{cluster}$.*

Proof. The existence of cluster set $P_{cluster}$ following from the fact that an infinite sequence in a compact space must have convergent sub-sequence. The sequence of probabilities

$(p_t^{AH}, p_t^{BL}, p_t^{BH})$ lives in a compact simplex

$$\Delta = \{(p^{AH}, p^{BL}, p^{BH}) | 0 \leq p^{AH}, p^{BL}, p^{BH} \leq 1; 0 \leq p^{AH} + p^{BL} + p^{BH} \leq 1\}.$$

The existence of a set of cluster points follows directly.

By the fact that p_s is a cluster point and the assumption that $\frac{p_t^{AH}}{p_t^{BH}} + \frac{p_t^{BL}}{p_t^{BH}} \to c$, we have $\frac{p_s^{AH}}{p_s^{BH}} + \frac{p_s^{BL}}{p_s^{BH}} = c$. This ratio is always well-defined for the reason that p_s^{BH} can't be 0. From the fact that $\frac{p_t^{AH}}{p_t^{BH}} + \frac{p_t^{BL}}{p_t^{BH}}$ forms a non-increasing sequence, if $p_s^{BH} = 0$, then $p_s^{AH} = p_s^{BL} = 0$. Furthermore, because of claim 18, $P_s = (0, 0, 0)$ is impossible. That $x_s > x_0$ follows directly from claims 17 and 18.

For a cluster point P_s and corresponding sub-sequence $P_{t_k^s}$, divide the sub-sequence further into two sub-sequences $P_{t_k^{s,a}}$ and $P_{t_k^{s,b}}$. Here at a particular belief $P_{t_k^s}$ if by the construction, action $\alpha_{t_k^s} = a$, then it is classified into $P_{t_k^{s,a}}$. Here at least one sub-sequence of $P_{t_k^{s,a}}$ and $P_{t_k^{s,b}}$ must be infinite. Without loss of generality, assume that $P_{t_k^{s,a}}$ is infinite. Then

$$P_{t_k^{s,a}+1} \to P_s(a).$$

By definition, $P_s(a)$ is a cluster point. ■

Following two claims says that, under condition 18, at a limit belief P_s, upon observing an action b, society's belief must no longer lives in the limit set $P_{cluster}$. In other words, under condition 18, from some period on, $\mathfrak{h}_t^{C_1}$ must solely consists of actions a.

Claim 20 *For each $P_s \in P_{cluster}$, we have*

$$\frac{p_s^{AH}}{p_s^{BL}} = \frac{\phi(b|BL, x_s) - \phi(b|BH, x_s)}{\phi(b|BH, x_s) - \phi(b|AH, x_s)} \tag{22}$$

Proof. By the fact that $\exists \alpha \in \{a, b\}$ such that $P_s(\alpha) \in P_{cluster}$, we have

$$\frac{p_s^{AH}}{p_s^{BH}} + \frac{p_s^{BL}}{p_s^{BH}} = \frac{p_s^{AH}}{p_s^{BH}} \frac{\phi(\alpha|AH, x_s)}{\phi(\alpha|BH, x_s)} + \frac{p_s^{BL}}{p_s^{BH}} \frac{\phi(\alpha|BL, x_s)}{\phi(\alpha|BH, x_s)},$$

which is equivalent to

$$p_s^{AH}[\phi(\alpha|BH, x_s) - \phi(\alpha|AH, x_s)] = p_s^{BL}[\phi(\alpha|BL, x_s) - \phi(\alpha|BH, x_s)]. \tag{23}$$

Following claim 17 and claim 18, $x_s > x_0$. We can verify that $\phi(\alpha|BH, x_s) - \phi(\alpha|AH, x_s) \neq 0$ and that $\phi(\alpha|BL, x_s) - \phi(\alpha|BH, x_s) \neq 0$. Lastly, that $p_s^{BL} = 0$ implies $p_s^{AH} = 0$, so

$\frac{p_s^{AH}}{p_s^{BH}} + \frac{p_s^{BL}}{p_s^{BH}} = c$ cannot hold. Therefore, we can rewrite equation 23 to obtain 22. ■

Claim 21 *If condition in lemma 16 is satisfied, then for all $P_s \in P_{cluster}$, $P_s(b) \notin P_{cluster}$*

Proof. Assume the opposite that $P_s(b) \in P_{cluster}$. Use the descripition 22, we have

$$\mathfrak{F}(x_s)\frac{\phi(b|AH, x_s)}{\phi(b|BL, x_s)} = \frac{\lambda_s^{AH}}{\lambda_s^{BL}}\frac{\phi(b|AH, x_s)}{\phi(b|BL, x_s)} = \frac{\phi(b|BL, x_s(b)) - \phi(b|BH, x_s(b))}{\phi(b|BH, x_s(b)) - \phi(b|AH, x_s(b))} = \mathfrak{F}(x_s(b)). \quad (24)$$

If $x_s \in (x_0, x_{BH})$, following the same reasoning as in formula 13, we have $x_s(b) > x_s$. By claim 33, we must have $\mathfrak{F}(x_s(b)) > \mathfrak{F}(x_s)$. By fact 35, $\frac{\phi(b|AH,x_s)}{\phi(b|BL,x_s)} < 1$ if $x_s \in (x_0, x_{BH})$. Therefore, if $x_s \in (x_0, x_{BH})$, we must have $\mathfrak{F}(x_s(b)) > \mathfrak{F}(x_s) > \mathfrak{F}(x_s)\frac{\phi(b|AH,x_s)}{\phi(b|BL,x_s)}$, which contradicts equation 24. So, if $x_s \in (x_0, x_{BH})$, then $P_s(b) \notin P_{cluster}$.

If $x_s \in [x_{BH}, 1]$, then by claim 32 and claim 33, we must have $\mathfrak{F}(x_s(b)) \geq \mathfrak{F}(y(x_s))$ where $y(x_s) = \frac{x_s\phi(b|BH,x_s)}{(1-x_s)\phi(b|AL,x_s)+x_s\phi(b|BH,x_s)}$. Then equation 24 contradicts the sufficient condition in lemma 16. ■

The following claim brings the contradiction: if $\mathfrak{h}_t^{C_1}$ consists of all action a from some period on, then no element in $P_{cluster}$ can actually be a limit point.

Claim 22 *If for all $P_s \in P_{cluster}$, $P_s(b) \notin P_{cluster}$. Then no P_s can be a limit point.*

Proof. For a cluster point P_s and a $P_{t_k^s}$ which is sufficiently close to P_s, by claim 21, $\alpha_{t_k^s}$ must be a.

If $\frac{p_s^{AH}}{p_s^{BH}} > 0$, then

$$\frac{p_{t_k^s+1}^{AH}}{p_{t_k^s+1}^{BH}} = \frac{p_{t_k^s}^{AH}}{p_{t_k^s}^{BH}}\frac{\phi(a|AH, x_{t_k^s})}{\phi(a|BH, x_{t_k^s})} > \frac{p_s^{AH}}{p_s^{BH}}.$$

Similarly, $P_{t_k^s+1}$ is sufficiently close to a different cluster point $P_s(a)$, $\alpha_{t_k^s+1}$ must be a as well. So $\frac{p_{t_k^s+2}^{AH}}{p_{t_k^s+2}^{BH}}$ must be even bigger.

Following this logic, P_t can never return within a neighborhood of P_s. This contradicts that P_s is a cluster point.

If $\frac{p_s^{AH}}{p_s^{BH}} = 0$, then $\frac{p_s^{BL}}{p_s^{BH}} = c > 0$. By claims 17 and 18, x_s must be strictly bigger than x_0. It is direct to verify that $\frac{\phi(a|BL,x_s)}{\phi(a|BH,x_s)} < 1$.Therefore,

$$\frac{p_{t_k^s+1}^{BL}}{p_{t_k^s+1}^{BH}} = \frac{p_{t_k^s}^{BL}}{p_{t_k^s}^{BH}}\frac{\phi(a|BL, x_{t_k^s})}{\phi(a|BH, x_{t_k^s})} < \frac{p_s^{BL}}{p_s^{BH}}.$$

So $\frac{p_t^{BL}}{p_t^{BH}}$ can never return to c. This implies that P_t can never return within a neighborhood of P_s again. ■ ■

Merely $\frac{\lambda_t^{AH}}{\lambda_t^{BH}} + \frac{\lambda_t^{BL}}{\lambda_t^{BH}} \to 0$ doesn't guarantee that Λ_t is eventually close to the axis λ^{BH}. The ratio could decrease to 0 just because that λ_t^{BH} increases much faster than λ_t^{AH} and λ_t^{BL}. We need to rule out this possibility.

Lemma 23 *If*
(1) $\lambda_t^{BH}(\mathfrak{h}_t^{C_1}|\Lambda) \to +\infty$*;*
or
(2) private signal is bounded ($\overline{s} < 1$*),* $\lambda_t^{BH}(\mathfrak{h}_t^{C_1}|\Lambda)$ *doesn't approach* $+\infty$*, but* $\exists \overline{t}$ *such that* $\lambda_t^{BH}(\mathfrak{h}_t^{C_1}|\Lambda) \geq \frac{\overline{s}}{1-\overline{s}}$ *for all* $t \geq \overline{t}$*.*
Then $\exists$ *sequence* $T_k \in \mathbb{N}$*, such that*

$$\lim_{T_k \to +\infty} \frac{\#a}{(T_k)^2} = f_a \equiv \frac{\log \frac{\phi(b|BL,1)}{\phi(b|AH,1)}}{\log \frac{\phi(a|AH,1)}{\phi(a|BL,1)} + \log \frac{\phi(b|BL,1)}{\phi(b|AH,1)}};$$
$$\lim_{T_k \to +\infty} \frac{\#b}{(T_k)^2} = f_b \equiv \frac{\log \frac{\phi(a|AH,1)}{\phi(a|BL,1)}}{\log \frac{\phi(a|AH,1)}{\phi(a|BL,1)} + \log \frac{\phi(b|BL,1)}{\phi(b|AH,1)}}. \tag{25}$$

Here $\#\alpha$ *counts the number of actions* $\alpha \in \{a, b\}$ *from period* T_k *to period* $T_k + (T_k)^2$*.*

Proof. If (1) holds, then $x_t(\mathfrak{h}_t^{C_1}|\Lambda) \to 1$. If (2) holds, then $x_t \geq \frac{1+\frac{\lambda_t^{BL}}{\lambda_t^{BH}}}{1+\frac{\lambda_t^{BL}}{\lambda_t^{BH}}+\frac{\lambda_t^{AH}}{\lambda_t^{BH}}+\frac{1-\overline{s}}{\overline{s}}}$ for all $t \geq \overline{t}$. Thus, $\liminf_{t\to+\infty} x_t \geq \overline{s}$. In both cases, $\forall k \in \mathbb{N}$, there exists T_k^1 such that $x_t \in (\overline{s} - \frac{1}{k}, 1]$ for all $t \geq T_k^1 - 1$.

We can verify that in phase I,

$$\mathfrak{h}_t^{C_1} = b \Leftrightarrow \frac{\lambda_t^{AH}}{\lambda_t^{BL}} > \mathfrak{F}(x_t). \tag{26}$$

Here $\mathfrak{F}(\cdot)$ is the same function as defined in lemma 16.

We have following claim: $\forall k \in \mathbb{N}$, $\exists T_k \geq T_k^1$ such that

$$\frac{\lambda_t^{AH}}{\lambda_t^{BL}} \in [\mathfrak{F}(\overline{s} - \frac{1}{k})\frac{\phi(b|AH, s_{*k}^b)}{\phi(b|BL, s_{*k}^b)}, \mathfrak{F}(1)\frac{\phi(a|AH, s_k^{*a})}{\phi(a|BL, s_k^{*a})}] \tag{27}$$

where $s_{*k}^\alpha = \text{argmin}_{x\in(\overline{s}-\frac{1}{k},1]} \frac{\phi(\alpha|AH,x)}{\phi(\alpha|BL,x)}$, and $s_k^{*\alpha} = \text{argmax}_{x\in(\overline{s}-\frac{1}{k},1]} \frac{\phi(\alpha|AH,x)}{\phi(\alpha|BL,x)}$. For notation convenience, from now to the end of this proof, we shall just write $[lb, ub]$ for the closed interval in 27.

In this paragraph we prove the above claim. Let $t_1 = \min\{t \geq T_k^1 | \frac{\lambda_t^{AH}}{\lambda_t^{BL}} > \mathfrak{F}(x_t)\}$. Then $t_1 < +\infty$. Otherwise, $\frac{\lambda_t^{AH}}{\lambda_t^{BH}} \leq \mathfrak{F}(x_t)$ for all $t \geq T_k^1$. By construction rule 26, we must have $\mathfrak{h}_t^{C_1} = a$ for all $t \geq T_k^1$. Then

$$\log \frac{\lambda_t^{AH}}{\lambda_t^{BL}} - \log \frac{\lambda_{T_k^1}^{AH}}{\lambda_{T_k+1}^{BL}} = \sum_{i=T_k}^{t-1} \log \frac{\phi(a|AH, x_i)}{\phi(a|BL, x_i)}.$$

By claim 35, $\log \frac{\phi(a|AH,x_i)}{\phi(a|BL,x_i)}$ is bounded above 0. Therefore, $\frac{\lambda_t^{AH}}{\lambda_t^{BL}} \to +\infty$ as $t \to +\infty$. However, this contradicts $\frac{\lambda_t^{AH}}{\lambda_t^{BL}} \leq \mathfrak{F}(x_t)$ for all $t \geq T_k^1$ since $\mathfrak{F}(x_t) \leq \mathfrak{F}(1) < +\infty$. (Recall $\mathfrak{F}(\cdot)$ strictly increases). Then we must have

$$\begin{aligned} \frac{\lambda_{t_1}^{AH}}{\lambda_{t_1}^{BL}} &= \frac{\lambda_{t_1-1}^{AH}}{\lambda_{t_1-1}^{BL}} \frac{\phi(a|AH, x_{t_1-1})}{\phi(a|BL, x_{t_1-1})} \\ &\leq \mathfrak{F}(x_{t_1-1}) \frac{\phi(a|AH, x_{t_1-1})}{\phi(a|BL, x_{t_1-1})} \\ &\leq \mathfrak{F}(1) \frac{\phi(a|AH, s_k^{*a})}{\phi(a|BL, s_k^{*a})} \end{aligned} \tag{28}$$

Here the first equation and the first inequality follow from the definition of t_1. The second inequality follows from that $x_{t_1-1} \in (\bar{s} - \frac{1}{k}, 1]$ and that $\mathfrak{F}(\cdot)$ strictly increases. Furthermore, we have

$$\begin{aligned} \frac{\lambda_{t_1}^{AH}}{\lambda_{t_1}^{BL}} &> \frac{\lambda_{t_1}^{AH}}{\lambda_{t_1}^{BL}} \frac{\phi(b|AH, x_{t_1})}{\phi(b|BL, x_{t_1})} \\ &\geq \mathfrak{F}(\bar{s} - \frac{1}{k}) \frac{\phi(b|AH, x_{t_1})}{\phi(b|BL, x_{t_1})} \\ &\geq \mathfrak{F}(\bar{s} - \frac{1}{k}) \frac{\phi(b|AH, s_{*k}^b)}{\phi(b|BL, s_{*k}^b)}. \end{aligned} \tag{29}$$

Here the first inequality follows from that $\frac{\phi(b|AH,x_{t_1})}{\phi(b|BL,x_{t_1})} < 1$ (see claim 35). The second inequality follows the definition of t_1 and that $\mathfrak{F}$ strictly increases. The third inequality follows from the definition of s_{*k}^b. Combine inequalities 28 and 29, we have $\frac{\lambda_{t_1}^{AH}}{\lambda_{t_1}^{BL}} \in [lb, ub]$. Let $T_k = t_1$. We have the following inductive argument: for all $t \geq T_k$, if $\frac{\lambda_t^{AH}}{\lambda_t^{BL}} \in [lb, ub]$, then $\frac{\lambda_{t+1}^{AH}}{\lambda_{t+1}^{BL}} \in [lb, ub]$. The inductive argument can be proved as following: there are two cases:

1. $\frac{\lambda_t^{AH}}{\lambda_t^{BL}} \leq x_t$, then $\frac{\lambda_{t+1}^{AH}}{\lambda_{t+1}^{BL}} = \frac{\lambda_t^{AH}}{\lambda_t^{BL}} \frac{\phi(a|AH,x_t)}{\phi(a|BL,x_t)} \leq \mathfrak{F}(1) \frac{\phi(a|AH,s_k^{*a})}{\phi(a|BL,s_k^{*a})}$.

2. $\frac{\lambda_t^{AH}}{\lambda_t^{BL}} > x_t$, then $\frac{\lambda_{t+1}^{AH}}{\lambda_{t+1}^{BL}} = \frac{\lambda_t^{AH}}{\lambda_t^{BL}} \frac{\phi(b|AH,x_t)}{\phi(b|BL,x_t)} > \mathfrak{F}(\bar{s} - \frac{1}{k}) \frac{\phi(b|AH,s_{*k}^b)}{\phi(b|BL,s_{*k}^a)}$.

So claim 27 is proved.

Furthermore, we have

$$\frac{\lambda_{T_k+(T_k)^2}^{AH}}{\lambda_{T_k+(T_k)^2}^{BL}} = \frac{\lambda_{T_k}^{AH}}{\lambda_{T_k}^{BL}} \Pi_{i=T_k}^{T_k+(T_k)^2-1} \frac{\phi(\alpha_i|AH, x_i)}{\phi(\alpha_i|BL, x_i)} \in [lb, ub]; \tag{30}$$

so

$$\Pi_{i=T_k}^{T_k+(T_k)^2-1} \frac{\phi(\alpha_i|AH, x_i)}{\phi(\alpha_i|BL, x_i)} \in [\frac{lb}{ub}, \frac{ub}{lb}]. \tag{31}$$

We can make left-hand side of 31 slightly bigger and obtain

$$\left((\frac{\phi(a|AH, s_k^{*a})}{\phi(a|BL, s_k^{*a})})^{\frac{\#a}{(T_k)^2}} (\frac{\phi(b|AH, s_k^{*b})}{\phi(b|BL, s_k^{*b})})^{\frac{\#b}{(T_k)^2}} \right)^{(T_k)^2} \geq \frac{lb}{ub}, \tag{32}$$

We can make left-hand side of 31 slightly smaller and obtain

$$\left((\frac{\phi(a|AH, s_{*k}^{a})}{\phi(a|BL, s_{*k}^{a})})^{\frac{\#a}{(T_k)^2}} (\frac{\phi(b|AH, s_{*k}^{b})}{\phi(b|BL, s_{*k}^{b})})^{\frac{\#b}{(T_k)^2}} \right)^{(T_k)^2} \leq \frac{ub}{lb}. \tag{33}$$

Now taking logarithm on both sides of 32 and 33, and let $k \to +\infty, T_k \to +\infty$, we have

$$\limsup_{T_k \to +\infty} \frac{\#a}{(T_k)^2} \log \frac{\phi(a|AH, 1)}{\phi(a|BL, 1)} + \big(1 - \limsup_{T_k \to +\infty} \frac{\#a}{(T_k^2)}\big) \log \frac{\phi(b|AH, 1)}{\phi(b|BL, 1)}) \leq 0; \tag{34}$$

and

$$\liminf_{T_k \to +\infty} \frac{\#a}{(T_k)^2} \log \frac{\phi(a|AH, 1)}{\phi(a|BL, 1)} + \big(1 - \liminf_{T_k \to +\infty} \frac{\#a}{(T_k^2)}\big) \log \frac{\phi(b|AH, 1)}{\phi(b|BL, 1)}) \geq 0. \tag{35}$$

Combine above two inequalities, we have

$$\liminf_{T_k \to +\infty} \frac{\#a}{(T_k)^2} \geq \frac{\log \frac{\phi(b|BL,1)}{\phi(b|AH,1)}}{\log \frac{\phi(a|AH,1)}{\phi(a|BL,1)} + \log \frac{\phi(b|BL,1)}{\phi(b|AH,1)}} \geq \limsup_{T_k \to +\infty} \frac{\#a}{(T_k)^2}.$$

So $\lim_{T_k \to +\infty} \frac{\#a}{(T_k)^2}$ exists and equals to $f_a \equiv \frac{\log \frac{\phi(b|BL,1)}{\phi(b|AH,1)}}{\log \frac{\phi(a|AH,1)}{\phi(a|BL,1)} + \log \frac{\phi(b|BL,1)}{\phi(b|AH,1)}}$. ■

Now we use above lemma to prove that we can always push society's belief sufficiently close to axis$-\lambda^{BH}$ in phase I.

Lemma 24 *In addition of sufficient condition 18, if*

$$\frac{\log \phi(a|AH,1) - \log \phi(a|BL,1)}{\log \phi(b|BL,1) - \log \phi(b|AH,1)} > \frac{\log \phi(a|AH,1) - \log \phi(a|AL,1)}{\log \phi(b|AL,1) - \log \phi(b|AH,1)}, \tag{36}$$

then there exists a sub-sequence t_k such that

$$\lambda_{t_k}^{AH} \to 0;\ \lambda_{t_k}^{BL} \to 0.$$

Proof. Assume the opposite. Then $\exists\ \varepsilon_0$ such that $\|(\lambda_t^{AH}, \lambda_t^{BL})\| > \varepsilon_0$ for sufficiently large t. Since we assumed sufficient condition 18, lemma 16 implies that $\frac{\lambda_t^{AH}}{\lambda_t^{BH}} + \frac{\lambda_t^{BL}}{\lambda_t^{BH}} \to 0$. Therefore we must have $\lambda_t^{BH} \to +\infty$. This is equivalent to $x_t \to 1$. This satisfies condition (1) in lemma 23.

Let $t_k = T_k + (T_k)^2$ where T_k as constructed in lemma 23. Then

$$\begin{aligned} \lambda_{T_k+(T_k)^2}^{AH} &= \lambda_{T_k}^{AH}\big[\Pi_{\alpha_t=a}\frac{\phi(a|AH,x_t)}{\phi(a|AL,x_t)}\big]\big[\Pi_{\alpha_t=b}\frac{\phi(b|AH,x_t)}{\phi(b|AL,x_t)}\big] \\ &\leq c^{T_k}\big[\Pi_{\alpha_t=a}\frac{\phi(a|AH,x_t)}{\phi(a|AL,x_t)}\big]\big[\Pi_{\alpha_t=b}\frac{\phi(b|AH,x_t)}{\phi(b|AL,x_t)}\big] \\ &\leq c^{T_k}\left[\big(\frac{\phi(a|AH,1)}{\phi(a|AL,1)}\big)^{\frac{\#a}{(T_k)^2}}\big(\frac{\phi(b|AH,1-\frac{1}{k})}{\phi(b|AL,1-\frac{1}{k})}\big)^{\frac{\#b}{(T_k)^2}}\right]^{(T_k)^2}. \end{aligned} \tag{37}$$

Here $c \equiv \max_{x\in[0,1]}\{\frac{\phi(a|AH,x)}{\phi(a|AL,x)}, \frac{\phi(b|AH,x)}{\phi(b|AL,x)}\}$ is the largest possible increase of λ^{AH}. Here the last inequality follows from (1) $x_t \in (1-\frac{1}{k}, 1]$ for $t \in [T_k, T_k+(T_k)^2]$, (2) for big enough k, $\frac{\phi(b|AH,x)}{\phi(b|AL,x)}$ monotonically decreases on $(1-\frac{1}{k}, 1)$ and (3) for big enough k, $\frac{\phi(a|AH,x)}{\phi(a|AL,x)}$ monotonically increases on $(1-\frac{1}{k}, 1)$. Condition 24 is equivalent to that $\big[\frac{\phi(a|AH,1)}{\phi(a|AL,1)}\big]^{f_a}\big[\frac{\phi(b|AH,1)}{\phi(b|AL,1)}\big]^{f_b} < 1$. So for sufficiently large T_k, the big term with the bracket in 37 is strictly below 1 and converges to $\big[\frac{\phi(a|AH,1)}{\phi(a|AL,1)}\big]^{f_a}\big[\frac{\phi(b|AH,1)}{\phi(b|AL,1)}\big]^{f_b}$. We have

$$\lim_{T_k\to\infty}\lambda_{T_k+(T_k)^2}^{AH} \leq \lim_{T_k\to\infty} c^{T_k}\big\{\big[\frac{\phi(a|AH,1)}{\phi(a|AL,1)}\big]^{f_a}\big[\frac{\phi(b|AH,1)}{\phi(b|AL,1)}\big]^{f_b}\big\}^{(T_k)^2}. \tag{38}$$

So $\lim_{T_k\to\infty}\lambda_{T_k+(T_k)^2}^{AH} = 0$. We can similarly prove $\lim_{T_k\to\infty}\lambda_{T_k+(T_k)^2}^{BL} = 0$. In fact, if $x_t \to 1$, then

$$\lim_{T_k\to\infty}\lambda_{T_k+(T_k)^2}^{BL} = \lim_{T_k\to\infty}\lambda_{T_k+(T_k)^2}^{AH}.$$

■

Recall that in phase II we use a long sequence of action b to push society's belief from a position close to axis-λ^{BH} to a $\varepsilon-$neighborhood of the confounded learning. As long as in phase II, $\lambda^{AH}, \lambda^{BL}$ stays negligible, the belief dynamics is similar to the one-dimension belief dynamics where $\lambda^{AH}, \lambda^{BH}$ are zero. In this sense, construction in phase I turns the problem from three-dimension into (roughly) one-dimension. However, we should notice that the (roughly) one-dimension dynamics in phase II is still different to a true one-dimension dynamics. We need to guarantee that $\lambda^{AH}, \lambda^{BL}$ stay negligible in entire phase II. To guarantee λ^{BL} stays negligible, we must start phase II with super small λ^{BL}. However, if $\lambda_t^{BH}(\mathfrak{h}_T^{C_1}|\Lambda) \to +\infty$, then this super small λ^{BL} comes with a cost of a super large λ^{BH}, and hence a super long sequence of actions b to reduce λ^{BH} close to π^*_{BH}. Since observing action b always increases λ^{BH}. It is not clear that whether the super small initial λ^{BL} can outweigh the super long sequence of actions b so that λ^{BL} stays negligible in phase II. We deal with this situation separately in proposition 31. If in phase I, we can arbitrarily shrink λ^{BL} without getting a super large λ^{BH}. Then it is easier to guarantee that $\frac{\lambda^{BL}}{\lambda^{BH}}$ stays small. After all, fix a learning environment, a λ^{BH} bounded from above implies that the number of actions b needed in phase II is also bounded from above. Hence the increase of $\frac{\lambda^{BL}}{\lambda^{BH}}$ in phase II is also bounded from above. Therefore, we can always control the largest value of $\frac{\lambda^{BL}}{\lambda^{BH}}$ in phase II by choosing a small enough initial value. In lemmas 27, 28 and proposition 30 we deal with this easier case.

From the next proposition to proposition 31, we all holds the following assumption:

Assumption 25 $\lambda^{BH}\frac{\phi(b|BH,\frac{\lambda^{BH}}{\lambda^{BH}+1})}{\phi(b|AL,\frac{\lambda^{BH}}{\lambda^{BH}+1})}$ *strictly increases in* λ^{BH} *on* $\lambda^{BH} \in (\frac{\underline{s}}{1-\underline{s}}, \frac{\overline{s}}{1-\overline{s}})$.

This assumption says that the belief updating rule of λ^{BH} is strictly increasing if $\lambda^{AH} = \lambda^{BL} = 0$. Then if λ^{BH} is above (below) π^*_{BH}, after seeing an action b, $\lambda^{BH}(b)$ cannot jump to the other side of π^*_{BH}, since π^*_{BH} is a fixed point of the belief updating rule. The following lemma generalizes this into the case that $\lambda^{AH}, \lambda^{BL}$ is negligible.

Lemma 26 *For any closed interval* $[\underline{b}, \overline{b}] \subset (\frac{\underline{s}}{1-\underline{s}}, \pi^*_{BH})$, *there exists* $\xi > 0$ *such that*

$$\Lambda \in [0,\xi]^2 \times [\underline{b}, \overline{b}] \Rightarrow \lambda^{BH}\frac{\phi(b|BH,\Lambda)}{\phi(b|AL,\Lambda)} \leq \pi^*_{BH}. \tag{39}$$

Similarly, for any closed interval $[\underline{b}, \overline{b}] \subset (\pi^*_{BH}, \frac{\overline{s}}{1-\overline{s}})$, *there exists* $\xi > 0$ *such that*

$$\Lambda \in [0,\xi]^2 \times [\underline{b}, \overline{b}] \Rightarrow \lambda^{BH}\frac{\phi(b|BH,\Lambda)}{\phi(b|AL,\Lambda)} \geq \pi^*_{BH} \tag{40}$$

.

Proof. We only write out the details of the case that $\Lambda \in [0,\xi]^2 \times [\underline{b}, \bar{b}] \subset [0,\xi]^2 \times (\frac{\underline{s}}{1-\underline{s}}, \pi^*_{BH})$. We compute the Taylor expansion of $\frac{\phi(b|BH,\Lambda)}{\phi(b|AL,\Lambda)}$ at $\overline{\Lambda} = (0,0,\lambda^{BH})$ with Lagrange remainder as following:

$$\begin{aligned}
&\frac{\phi(b|BH,\Lambda)}{\phi(b|AL,\Lambda)} - \frac{\phi(b|BH,\overline{\Lambda})}{\phi(b|AL,\overline{\Lambda})} \\
= \quad & (\lambda^{AH}, \lambda^{BL}, 0) \begin{bmatrix} \frac{\partial}{\partial \lambda^{AH}} \frac{\phi(b|BH,\Lambda)}{\phi(b|AL,\Lambda)}|_{\overline{\Lambda}} \\ \frac{\partial}{\partial \lambda^{BL}} \frac{\phi(b|BH,\Lambda)}{\phi(b|AL,\Lambda)}|_{\overline{\Lambda}} \\ \frac{\partial}{\partial \lambda^{BH}} \frac{\phi(b|BH,\Lambda)}{\phi(b|AL,\Lambda)}|_{\overline{\Lambda}} \end{bmatrix} + (\lambda^{AH}, \lambda^{BL}, 0) H|_{\tilde{\Lambda}} \begin{bmatrix} \lambda^{AH} \\ \lambda^{BL} \\ 0 \end{bmatrix} \\
= \quad & \frac{-\lambda^{BH}\lambda^{AH} + \lambda^{BL}}{(1+\lambda^{BH})^2} \frac{\partial}{\partial x} \frac{\phi(b|BH,x)}{\phi(b|AL,x)}|_{\overline{x}} + (\lambda^{AH}, \lambda^{BL}, 0) H|_{\tilde{\Lambda}} \begin{bmatrix} \lambda^{AH} \\ \lambda^{BL} \\ 0 \end{bmatrix}.
\end{aligned}$$

Here $\tilde{\Lambda} = c(\Lambda - \overline{\Lambda}) + \overline{\Lambda}, 0 < c < 1$ and $\overline{x} = \frac{\lambda^{BH}}{1+\lambda^{BH}}$. We can verify that $\tilde{\Lambda} \in [0,\xi]^2 \times [\underline{b}, \bar{b}]$ and $\overline{x} \in [\frac{\underline{b}}{\underline{b}+1}, \frac{\bar{b}}{\bar{b}+1}] \subset (\underline{s}, x_{BH})$.

With assumption that $F^A(s), F^B(s)$ are twice continuously differentiable on $(\underline{s}, \overline{s})$ (see assumption 1), we have that $\frac{\partial}{\partial x} \frac{\phi(b|BH,x)}{\phi(b|AL,x)}|_{\overline{x}}$ is continuous in $\overline{x}$ on $[\frac{\underline{b}}{1+\underline{b}}, \frac{\bar{b}}{\bar{b}+1}]$. Furthermore, for all $\tilde{\Lambda} = (c\lambda^{AH}, c\lambda^{BL}, \lambda^{BH})$, we have that $\tilde{x} \equiv \frac{\lambda^{BH}+c\lambda^{BL}}{1+c\lambda^{AH}+c\lambda^{BL}+\lambda^{BH}} \geq \frac{\lambda^{BH}}{1+\lambda^{BH}+\xi} \geq \frac{\underline{b}}{\underline{b}+1+\xi}$, and that $\tilde{x} \leq \frac{\lambda^{BH}+\xi}{1+\lambda^{BH}+\xi} \leq \frac{\bar{b}+\xi}{1+\bar{b}+\xi}$. By choosing $\xi < \min\{\frac{1-\underline{s}}{\underline{s}}\underline{d} - 1, \pi_{BH} - \bar{b}\}$, we can guarantee that $\tilde{x} \in (\underline{s}, x_{BH})$. Thus H_{ij} is continuous on $\tilde{\Lambda} \in [0,\xi]^2 \times [\underline{b}, \bar{b}]$.

Let $\overline{M} = \text{argmax}_{x \in [\frac{\underline{b}}{\underline{b}+1}, \frac{\bar{b}}{\bar{b}+1}]} \frac{\partial}{\partial x} \frac{\phi(b|BH,x)}{\phi(b|AL,x)}|_{\overline{x}}$, $\underline{M} = \text{argmin}_{x \in [\frac{\underline{b}}{\underline{b}+1}, \frac{\bar{b}}{\bar{b}+1}]} \frac{\partial}{\partial x} \frac{\phi(b|BH,x)}{\phi(b|AL,x)}|_{\overline{x}}$ and $N = \text{argmax}_{\tilde{\Lambda} \in [0,\xi]^2 \times [\underline{b}, \bar{b}]} H_{ij}$. We observe that $-\frac{\bar{b}}{(1+\bar{b})(1+\underline{b})}\xi \leq \frac{-\lambda^{BH}\lambda^{AH}+\lambda^{BL}}{(1+\lambda^{BH})^2} \leq \frac{1}{(1+\underline{b})^2}\xi$.Then we have

$$\frac{\phi(b|BH,\Lambda)}{\phi(b|AL,\Lambda)} - \frac{\phi(b|BH,\overline{\Lambda})}{\phi(b|AL,\overline{\Lambda})} \leq \max\{\frac{\overline{M}}{(1+\underline{b})^2}, -\frac{\underline{M}\bar{b}}{(1+\bar{b})(1+\underline{b})}\}\xi + \max\{4N, 0\}\xi^2.$$

Furthermore, if $\Lambda \in [0,\xi]^2 \times [\underline{b},\bar{b}]$, we have

$$\begin{aligned} & \lambda^{BH}\frac{\phi(b|BH,\Lambda)}{\phi(b|AL,\Lambda)} \\ \leq\ & \lambda^{BH}\frac{\phi(b|BH,\frac{\lambda^{BH}}{\lambda^{BH}+1})}{\phi(b|AL,\frac{\lambda^{BH}}{\lambda^{BH}+1})} + \lambda^{BH}\Big(\max\{\frac{\overline{M}}{(1+\underline{b})^2}, -\frac{\underline{M}\bar{b}}{(1+\bar{b})(1+\underline{b})}\}\xi + \max\{4N,0\}\xi^2\Big) \\ \leq\ & \bar{b}\frac{\phi(b|BH,\frac{\bar{b}}{\bar{b}+1})}{\phi(b|AL,\frac{\bar{b}}{\bar{b}+1})} + \bar{b}\Big(\max\{\frac{\overline{M}}{(1+\underline{b})^2}, -\frac{\underline{M}\bar{b}}{(1+\bar{b})(1+\underline{b})}\}\xi + \max\{4N,0\}\xi^2\Big) \end{aligned}$$

which is smaller than π^*_{BH} for small enough ξ. ■

The following lemma says: if society's current belief Λ is sufficiently close to axis-λ^{BH}, and λ^{BH} is somewhere between $\underline{s}$ and π^*_{BH}, then a sequence of actions b can push the society's belief into the $\varepsilon-$neighborhood.

Lemma 27 *With assumption 25, if*

$$\forall\gamma > 0, \exists\Lambda_{T_\gamma} s.t. x_{T_\gamma} > x_0, \lambda^{AH}_{T_\gamma} < \gamma, \lambda^{BL}_{T_\gamma} < \gamma, \lambda^{BH}_{T_\gamma} \leq \pi^*_{BH} - \varepsilon/2 \tag{41}$$

then there exists a γ_0 *and* t_0 *such that*

$$\lambda^{AH}_{T_{\gamma_0}+t_0}(\{b\}^{t_0}|\Lambda_{T_{\gamma_0}}) < \frac{\varepsilon}{2}, \lambda^{BL}_{T_{\gamma_0}+t_0}(\{b\}^{t_0}|\Lambda_{T_{\gamma_0}}) < \frac{\varepsilon}{2}, \lambda^{BH}_{T_{\gamma_0}+t_0}(\{b\}^{t_0}|\Lambda_{T_{\gamma_0}}) \in (\pi_{BH} - \frac{\varepsilon}{2}, \pi^*_{BH}].$$

In other words, if we can push society's belief arbitrarily close to axis$-\lambda^{BH}$ while keeping λ^{BH} below π^*_{BH}, then we can always push the society's belief to a proper position, from where t_0 actions b leads society's belief into the $\varepsilon-$neighborhood.

Before the formal proof, let us try to explain the intuition that the lemma is true. Recall that $x = \frac{\lambda^{BH}+\lambda^{BL}}{1+\lambda^{AH}+\lambda^{BL}+\lambda^{BH}}$. After an algebraic transformation, we can see that $x_{T_\gamma} > x_0$ actually put a lower bound on $\lambda^{BH}_{T_\gamma}$, if $\lambda^{AH}, \lambda^{BL} < \gamma$ for a super small γ. The smaller the γ is, the closer the lower bound is to $\frac{x_0}{1-x_0}$. We can roughly think λ^{BH}'s starting position is within the closed interval $[(1-d)\frac{x_0}{1-x_0}, \pi^*_{BH} - \varepsilon/2]$ for a d close to 1.

Now, conditional on seeing an action b, λ^{BH} moves up. If λ^{BH} hasn't moved above $\pi^*_{BH} - \varepsilon/2$, and $\lambda^{AH}, \lambda^{BL}$ stays negligible, then Λ is bounded away from confounded learning Λ^*, hence each observation of b multiplies λ^{BH} by a number bounded below by $\eta > 1$. Therefore, the number of actions b needed to push λ^{BH} above $\pi^*_{BH} - \varepsilon/2$ is bounded above by a finite number $\overline{N}$.

Because $\frac{\phi(b|BL,x)}{\phi(b|AL,x)}$ is bounded above as well (since it is continuous in x on a closed interval), within the $\overline{N}$ steps, λ^{BL}'s increases is also bounded above. Therefore, by starting with a sufficiently small $\lambda^{BL}_{T_\gamma}$, we can obtain λ^{BL} stays negligible until λ^{BH} moves above $\pi^*_{BH} - \varepsilon/2$. We automatically obtain that λ^{AH} stays negligible to λ^{BH} since $\frac{\phi(b|AH,x)}{\phi(b|BH,x)} < 1$ always holds.

Lastly, by keeping $\lambda^{AH}, \lambda^{BL}$ small enough, λ^{BH} cannot move above π^*_{BH} because of the monotonicity assumption 25. So we obtain the conclusion of this lemma.

Proof.

For each Λ_{T_γ}, we construct an auxilliary process $\tilde{\Lambda}$ as following:

$$\begin{aligned}
\tilde{\Lambda}_{T_\gamma} &= \Lambda_{T_\gamma} \\
\tilde{\lambda}^{BH}_{t+1} &= \tilde{\lambda}^{BH}_t \frac{\phi(b|BH,\underline{x}^{down})}{\phi(b|AL,\underline{x}^{down})}, \forall t \geq T_\gamma; \\
\frac{\tilde{\lambda}^{BL}_{t+1}}{\tilde{\lambda}^{BH}_{t+1}} &= \frac{\tilde{\lambda}^{BL}_t}{\tilde{\lambda}^{BH}_t} \frac{\phi(b|BL,\overline{x}^{down})}{\phi(b|BH,\overline{x}^{down})}, \forall t \geq T_\gamma; \\
\frac{\tilde{\lambda}^{AH}_{t+1}}{\tilde{\lambda}^{BH}_{t+1}} &= \frac{\tilde{\lambda}^{AH}_t}{\tilde{\lambda}^{BH}_t} \frac{\phi(b|AH,1)}{\phi(b|BH,1)}, \forall t \geq T_\gamma.
\end{aligned}$$

Here $\overline{x}^{down} = \text{argmax}_{x\in[x_0, x_{BH}-\delta^{down}]} \frac{\phi(b|BL,x)}{\phi(b|BH,x)}$, $\underline{x}^{down} = \text{argmin}_{x\in[x_0, x_{BH}-\delta^{down}]} \frac{\phi(b|BH,x)}{\phi(b|AL,x)}$, and δ^{down} is a small positive number defined in claim 37. This auxiliary process is constructed with the purpose that $\frac{\tilde{\lambda}^{BL}_t}{\tilde{\lambda}^{BH}_t} \geq \frac{\lambda^{BL}_t}{\lambda^{BH}_t}$ and $\tilde{\lambda}^{BH}_t \leq \lambda^{BH}_t$. In this way, we could use auxiliary values $\tilde{\lambda}^{BL}$, $\tilde{\lambda}^{BH}$ to control the real values λ^{BL} and λ^{BH}.

We have the following claim: $\forall c^{down}_E \in (0, \frac{\varepsilon/2}{\pi^*_{BH}-\varepsilon/2})$ *and* $\forall d \in (0,1)$, $\exists \gamma_0 > 0$ *and* t_1 *such that*

$$\lambda^{BH}_{T_{\gamma_0}} > (1-d)\frac{x_0}{1-x_0}; \frac{\lambda^{AH}_{T_{\gamma_0}}}{\lambda^{BH}_{T_{\gamma_0}}} < c^{down}_E; \tilde{\lambda}^{BH}_{T_{\gamma_0}+t_1} > \pi_{BH} - \varepsilon/2; \quad \frac{\tilde{\lambda}^{BL}_{T_{\gamma_0}+t_1}}{\tilde{\lambda}^{BH}_{T_{\gamma_0}+t_1}} < c^{down}_E. \tag{42}$$

In this paragraph we prove the above claim. First, we can verify that $x_{T_\gamma} > x_0$ implies that $\lambda^{BH}_{T_\gamma} > -\gamma + \frac{x_0}{1-x_0}$. By choosing $\gamma < \min\{d\frac{x_0}{1-x_0}, c^{down}_E(1-d)\frac{x_0}{1-x_0}\}$, we can have $\lambda^{BH}_{T_\gamma} > (1-d)\frac{x_0}{1-x_0}$ and $\lambda^{AH}_{T_\gamma} < \gamma < c^{down}_E(1-d)\frac{x_0}{1-x_0}$. Then $\frac{\lambda^{AH}_{T_\gamma}}{\lambda^{BH}_{T_\gamma}} < c^{down}_E$. Second, that $\exists t_1$ such that

$$\tilde{\lambda}^{BH}_{T_{\gamma_0}+t_1} > \pi_{BH} - \varepsilon/2; \quad \frac{\tilde{\lambda}^{BL}_{T_{\gamma_0}+t_1}}{\tilde{\lambda}^{BH}_{T_{\gamma_0}+t_1}} < c^{down}_E. \tag{43}$$

This is equivalent to: $\exists t_1$ such that

$$\tilde{\lambda}^{BH}_{T_\gamma}\Big(\frac{\phi(b|BH,\underline{x}^{down})}{\phi(b|AL,\underline{x}^{down})}\Big)^{t_1} > \pi^*_{BH}-\varepsilon/2;$$
$$\frac{\tilde{\lambda}^{BL}_{T_\gamma}}{\tilde{\lambda}^{BH}_{T_\gamma}}\Big(\frac{\phi(b|BL,\overline{x}^{down})}{\phi(b|BH,\overline{x}^{down})}\Big)^{t_1} < c^{down}_E;$$

which is further equivalent to

$$\frac{\log c^{down}_E-\log\lambda^{BL}_{T_\gamma}}{\log\frac{\phi(b|BL,\overline{x}^{down})}{\phi(b|BH,\overline{x}^{down})}}+\log\lambda^{BH}_{T_\gamma}\Big[\frac{1}{\log\frac{\phi(b|BL,\overline{x}^{down})}{\phi(b|BH,\overline{x}^{down})}}+\frac{1}{\log\frac{\phi(b|BH,\underline{x}^{down})}{\phi(b|AL,\underline{x}^{down})}}\Big]-\frac{\log(\pi_{BH}-\varepsilon/2)}{\log\frac{\phi(b|BH,\underline{x}^{down})}{\phi(b|AL,\underline{x}^{d}own)}}>1.(44)$$

Since $\lambda^{BH}_{T_\gamma}\in((1-d)\frac{x_0}{1-x_0},\pi^*_{BH}-\varepsilon/2)$, as γ decreases, the left-hand side of 44 increases to $+\infty$, so t_1 certainly exists. From here to the end of this proof, let's choose a $\gamma_0(c^{down}_E)$ for each $c^{down}_E\in(0,\frac{\varepsilon/2}{\pi^*_{BH}-\varepsilon/2})$. For notation convenience, we write γ_0 for $\gamma_0(c^{down}_E)$.

Intuitively, as $\tilde{\lambda}^{BH}$ increases slower than λ^{BH}, λ^{BH} must move above $\pi^*_{BH}-\varepsilon/2$ before period t_1. We claim this intuition is true: $\exists t\in\{0,1,\ldots,t_1\}$ such that $\lambda^{BH}_t>\pi^*_{BH}-\varepsilon/2$.

In this paragraph, we prove the above claim. Let us use I_{t_1} as an abbreviation of index set $\{0,1,\ldots,t_1\}$. We first assume that $\forall t\in I_{t_1}$, $\lambda^{BH}_t\le\pi^*_{BH}-\varepsilon/2$. Under this assumption, we have following inductive argument: $\forall t\in I_{t_1}-\{t_1\}$, if

$$\frac{\tilde{\lambda}^{BL}_{T_{\gamma_0}+t}}{\tilde{\lambda}^{BH}_{T_{\gamma_0}+t}}\ge\frac{\lambda^{BL}_{T_{\gamma_0}+t}}{\lambda^{BH}_{T_{\gamma_0}+t}};\ \tilde{\lambda}^{BH}_{T_{\gamma_0}+t}\le\lambda^{BH}_{T_{\gamma_0}+t};\ x_{T_{\gamma_0}+t}\in[x_0,x_{BH}-\delta^{down}] \tag{45}$$

then

$$\frac{\tilde{\lambda}^{BL}_{T_{\gamma_0}+t+1}}{\tilde{\lambda}^{BH}_{T_{\gamma_0}+t+1}}\ge\frac{\lambda^{BL}_{T_{\gamma_0}+t+1}}{\lambda^{BH}_{T_{\gamma_0}+t+1}};\ \tilde{\lambda}^{BH}_{T_{\gamma_0}+t+1}\le\lambda^{BH}_{T_{\gamma_0}+t+1};\ x_{T_{\gamma_0}+t+1}\in[x_0,x_{BH}-\delta^{down}] \tag{46}$$

The proof for the inductive argument is as following:

$$\frac{\tilde{\lambda}^{BL}_{T_{\gamma_0}+t+1}}{\tilde{\lambda}^{BH}_{T_{\gamma_0}+t+1}}=\frac{\tilde{\lambda}^{BL}_{T_{\gamma_0}+t}}{\tilde{\lambda}^{BH}_{T_{\gamma_0}+t}}\frac{\phi(b|BH,\overline{x}^{down})}{\phi(b|AL,\overline{x}^{down})}\ge\frac{\lambda^{BL}_{T_{\gamma_0}+t}}{\lambda^{BH}_{T_{\gamma_0}+t}}\frac{\phi(b|BH,x_{T_{\gamma_0}+t})}{\phi(b|AL,x_{T_{\gamma_0}+t})}. \tag{47}$$

Here the inequality follows inductive assumption $\frac{\tilde{\lambda}^{BL}_{T_{\gamma_0}+t}}{\tilde{\lambda}^{BH}_{T_{\gamma_0}+t}}\ge\frac{\lambda^{BL}_{T_{\gamma_0}+t}}{\lambda^{BH}_{T_{\gamma_0}+t}}$, $x_{T_{\gamma_0}+t}\in[x_0,x_{BH}-\delta^{down}]$ and the definition that $\overline{x}^{down}=\text{argmax}_{x\in[x_0,x_{BH}-\delta^{down}]}\frac{\phi(b|BL,x)}{\phi(b|BH,x)}$. The proof for $\tilde{\lambda}^{BH}_{T_{\gamma_0}+t+1}\le$

$\lambda^{BH}_{T_{\gamma_0}+t+1}$ is similar. By assumption, $\lambda^{BH}_{T_{\gamma_0}+t+1} \leq \pi^*_{BH} - \varepsilon/2$. From claim 42 and the definition of $\frac{\tilde{\lambda}^{BL}}{\tilde{\lambda}^{BH}}$, we have $\frac{\tilde{\lambda}^{BL}_{T_{\gamma_0}+t+1}}{\tilde{\lambda}^{BH}_{T_{\gamma_0}+t+1}} < \frac{\tilde{\lambda}^{BL}_{T_{\gamma_0}+t_1}}{\tilde{\lambda}^{BH}_{T_{\gamma_0}+t_1}} < c^{down}_E$. Therefore, $x_{T_{\gamma_0}+t+1} \leq x_{BH} - \delta^{down}$ following claim 37. Finally, rewrite $x_{T_{\gamma_0}+t+1} = \frac{1}{1+\frac{1}{\frac{\lambda^{BH}_{T_{\gamma_0}+t+1}+\lambda^{BL}_{T_{\gamma_0}+t+1}}{1+\lambda^{AH}_{T_{\gamma_0}+t+1}}}}$. Then following inductive assumption that $x_{T_{\gamma_0}+t} \in [x_0, x_{BH} - \delta^{down}]$ and the reasoning in 13, we have $x_{T_{\gamma_0}+t+1} \geq x_{T_{\gamma_0}+t}$. So $x_{T_{\gamma_0}+t+1} \in [x_0, x_{BH} - \delta^{down}]$. We also verify that inductive assumption holds for $t = 0$.

Following the inductive proof, we must have $\tilde{\lambda}^{BH}_{T_{\gamma_0}+t_1} \leq \lambda^{BH}_{T_{\gamma_0}+t_1}$. However, in claim 42 we have $\tilde{\lambda}^{BH}_{T_{\gamma_0}+t_1} > \pi^*_{BH} - \varepsilon/2$. This contradicts the assumption that $\lambda^{BH}_{T_\gamma+t} \leq \pi^*_{BH} - \varepsilon/2$ for all $t \in I_{t_1}$.

Now let $t_0 = \min\{t | \lambda^{BH}_{T_{\gamma_0}+t} > \pi^*_{BH} - \varepsilon/2\}$. Then $\lambda^{BH}_{T_{\gamma_0}+t} \leq \pi^*_{BH} - \varepsilon/2$ for all $t \in \{0, 1, \dots, t_0 - 1\}$. The above inductive argument still works for $t \in \{0, \dots, t_0 - 2\}$. Therefore, we have that $c^{down}_E > \frac{\tilde{\lambda}^{BL}_{T_{\gamma_0}+t_0-1}}{\tilde{\lambda}^{BH}_{T_{\gamma_0}+t_0-1}} > \frac{\lambda^{BL}_{T_{\gamma_0}+t_0-1}}{\lambda^{BH}_{T_{\gamma_0}+t_0-1}}$ and that $\lambda^{BH}_{T_{\gamma_0}+t_0-1} \leq \pi^*_{BH} - \varepsilon/2$. Furthermore, since observing action b always reduces $\frac{\lambda^{AH}}{\lambda^{BH}}$, $\frac{\lambda^{AH}_{T_{\gamma_0}+t_0-1}}{\lambda^{BH}_{T_{\gamma_0}+t_0-1}} < \frac{\lambda^{AH}_{T_{\gamma_0}}}{\lambda^{BH}_{T_{\gamma_0}}} < c^{down}_E$. Therefore, $\lambda^{AH}_{T_{\gamma_0}+t_0-1}, \lambda^{BL}_{T_{\gamma_0}+t_0-1} < c^{down}_E(\pi^*_{BH} - \varepsilon/2)$.

To summarize, up to this point, we have proved that: $\forall c^{down}_E \in (0, \frac{\varepsilon/2}{\pi^*_{BH}-\varepsilon/2})$ *and* $\forall d \in (0,1)$, $\exists\ \gamma_0 \in (0, \min\{d\frac{x_0}{1-x_0}, c^{down}_E(1-d)\frac{x_0}{1-x_0}\})$ *and* $t_0(\gamma_0, T_{\gamma_0})$ *such that*

$$\Lambda_{T_{\gamma_0}+t_0-1} \in [0, c^{down}_E(\pi^*_{BH} - \varepsilon/2)]^2 \times [(1-d)\frac{x_0}{1-x_0}, \pi^*_{BH} - \varepsilon/2]. \tag{48}$$

By choosing d small enough, we have $[(1-d)\frac{x_0}{1-x_0}, \pi^*_{BH} - \varepsilon/2] \in (\frac{\underline{s}}{1-\underline{s}}, \pi^*_{BH})$. (Recall $x_0 > \underline{s}$ is necessary for learning). Following lemma 26, we can find a c^{down}_E small enough such that $\lambda^{BH}_{T+t_0} \leq \pi^*_{BH}$. Therefore, $\lambda^{BH}_{T_{\gamma_0}+t_0} \in (\pi^*_{BH} - \varepsilon/2, \pi^*_{BH}]$. Furthermore, we have $\frac{\lambda^{AH}_{T_{\gamma_0}+t_0}}{\lambda^{BH}_{T_{\gamma_0}+t_0}} < \frac{\lambda^{AH}_{T_{\gamma_0}}}{\lambda^{BH}_{T_{\gamma_0}}} < c^{down}_E$. So $\lambda^{AH}_{T_{\gamma_0}+t_0} \leq \varepsilon/2$ as long as $c^{down}_E < \frac{\varepsilon/2}{\pi^*_{BH}}$. Finally, $\frac{\lambda^{BL}_{T_{\gamma_0}+t_0}}{\lambda^{BH}_{T_{\gamma_0}+t_0}} = \frac{\lambda^{BL}_{T_{\gamma_0}+t_0-1}}{\lambda^{BH}_{T_{\gamma_0}+t_0-1}} \frac{\phi(b|BH, x_{T_{\gamma_0}+t_0-1})}{\phi(b|AL, x_{T_{\gamma_0}+t_0-1})} < c^{down}_E \frac{\phi(b|BH,x_0)}{\phi(b|AL,x_0)}$. Here the last inequality following from that $\frac{\phi(b|BH,x)}{\phi(b|AL,x)}$ monotonically decreases on (x_0, x_{BH}). (See result 2 in claim 38). Therefore, $\lambda^{BL}_{T_{\gamma_0}+t_0} < \varepsilon/2$ as long as $c^{down}_E < \frac{\varepsilon/2}{\pi^*_{BH}} \frac{\phi(b|BH,x_0)}{\phi(b|AL,x_0)}$.

To summarize, we use c^{down}_E to control the largest possible value for $\lambda^{AH}, \lambda^{BL}$ in phase II. As long as c^{down}_E is small enough, λ^{BH} must increase but cannot jump above π^*_{BH}, after seeing a long sequence of action b. Furthermore, by choosing γ_0 sufficiently smaller than c^{down}_E we can guarantee $\lambda^{AH}, \lambda^{BL} < c^{down}_E$ in phase II. ■

Then next lemma is very similar to the previous lemma. The only difference is that we approach the confounded learning from above.

Lemma 28 *With assumption 25, if*

$$\forall \gamma > 0, \exists \Lambda_{T_\gamma}\ s.t.\ \lambda^{AH}_{T_\gamma} < \gamma, \lambda^{BL}_{T_\gamma} < \gamma, \lambda^{BH}_{T_\gamma} \in [\pi^*_{BH} + \varepsilon/2, \overline{\lambda}^{BH}] \subset (\pi^*_{BH}, \frac{\overline{s}}{1-\overline{s}}) \tag{49}$$

then there exists a γ_0 *and* t_0 *such that*

$$\lambda^{AH}_{T_{\gamma_0}+t_0}(\{b\}^{t_0}|\Lambda_{T_{\gamma_0}}) < \frac{\varepsilon}{2}, \lambda^{BL}_{T_{\gamma_0}+t_0}(\{b\}^{t_0}|\Lambda_{T_{\gamma_0}}) < \frac{\varepsilon}{2}, \lambda^{BH}_{T_{\gamma_0}+t_0}(\{b\}^{t_0}|\Lambda_{T_{\gamma_0}}) \in [\pi_{BH}, \pi^*_{BH} + \frac{\varepsilon}{2}).$$

In other words, if we can push society's belief arbitrarily close to axis$-\lambda^{BH}$ *while keeping* λ^{BH} *above* π^*_{BH}*, then we can always push the society's belief to a proper position, from where* t_0 *actions b leads society's belief into the* $\varepsilon-$*neighborhood.*

Proof. For each Λ_{T_γ}, we construct an auxilliary process $\tilde{\Lambda}$ as following:

$$\begin{aligned}
\tilde{\Lambda}_{T_\gamma} &= \Lambda_{T_\gamma} \\
\tilde{\lambda}^{BH}_{t+1} &= \tilde{\lambda}^{BH}_t \frac{\phi(b|BH, \underline{x}^{up})}{\phi(b|AL, \underline{x}^{up})}, \forall t \geq T_\gamma; \\
\frac{\tilde{\lambda}^{BL}_{t+1}}{\tilde{\lambda}^{BH}_{t+1}} &= \frac{\tilde{\lambda}^{BL}_t}{\tilde{\lambda}^{BH}_t} \frac{\phi(b|BL, \overline{x}^{up})}{\phi(b|BH, \overline{x}^{up})}, \forall t \geq T_\gamma; \\
\frac{\tilde{\lambda}^{AH}_{t+1}}{\tilde{\lambda}^{BH}_{t+1}} &= \frac{\tilde{\lambda}^{AH}_t}{\tilde{\lambda}^{BH}_t} \frac{\phi(b|AH, 1)}{\phi(b|BH, 1)}, \forall t \geq T_\gamma.
\end{aligned}$$

Here $\overline{x}^{up} = \text{argmax}_{x \in [x_{BH}+\delta^{up}, 1]} \frac{\phi(b|BL,x)}{\phi(b|BH,x)}$, $\underline{x}^{up} = \text{argmax}_{x \in [x_{BH}+\delta^{up}, 1]} \frac{\phi(b|BH,x)}{\phi(b|AL,x)}$, and δ^{up} is a small positive number defined in claim 36.

We have the following claim: $\forall c^{up}_E \in (0, \min\{\frac{\varepsilon/2}{\pi_{BH}(\pi_{BH}+\varepsilon/2)}, \frac{\varepsilon/2}{\pi_{BH}+\varepsilon/2}\})$, $\exists \gamma_0 > 0$ *and* t_1 *such that*

$$\frac{\lambda^{AH}_{T_{\gamma_0}}}{\lambda^{BH}_{T_{\gamma_0}}} < c^{up}_E; \tilde{\lambda}^{BH}_{T_{\gamma_0}+t_1} < \pi^*_{BH} + \varepsilon/2; \quad \frac{\tilde{\lambda}^{BL}_{T_{\gamma_0}+t_1}}{\tilde{\lambda}^{BH}_{T_{\gamma_0}+t_1}} < c^{up}_E. \tag{50}$$

In this paragraph we prove the above claim. First, by choosing $\gamma < c^{up}_E(\pi^*_{BH} + \varepsilon/2)$, we can have $\frac{\lambda^{AH}_{T_\gamma}}{\lambda^{BH}_{T_\gamma}} < c^{up}_E$. Second, we can verify the existence of t_1 is equivalent to

$$\frac{\log c^{up}_E - \log \lambda^{BH}_{T_\gamma}}{\log \frac{\phi(b|BL,\overline{x}^{up})}{\phi(b|BH,\overline{x}^{up})}} + \log \lambda^{BH}_{T_\gamma} \left(\frac{1}{\log \frac{\phi(b|BL,\overline{x}^{up})}{\phi(b|BH,\overline{x}^{up})}} + \frac{1}{\log \frac{\phi(b|BH,\underline{x}^{up})}{\phi(b|AL,\underline{x}^{up})}} \right) - \frac{\log(\pi^*_{BH} + \varepsilon/2)}{\log \frac{\phi(b|BH,\underline{x}^{up})}{\phi(b|AL,\underline{x}^{u})}} > 1 \tag{51}$$

Since $\lambda_{T_\gamma}^{BH} \in (\pi_{BH}^* + \varepsilon/2, \overline{\lambda}^{BH})$, as γ decreases, the left-hand side of 51 increases to $+\infty$, so t_1 certainly exists. From here to the end of this proof, let's choose a $\gamma_0(c_E^{up})$ for each $c_E^{up} \in (0, \min\{\frac{\varepsilon/2}{\pi_{BH}(\pi_{BH}+\varepsilon/2)}, \frac{\varepsilon/2}{\pi_{BH}+\varepsilon/2}\})$. For notation convenience, we write γ_0 for $\gamma_0(c_E^{up})$.

We claim this intuition is true: $\exists t \in \{0, 1, \ldots, t_1\}$ such that $\lambda_t^{BH} < \pi_{BH}^* + \varepsilon/2$.

In this paragraph, we prove above claim. Let us use I_{t_1} as an abbreviation of index set $\{0, 1, \ldots, t_1\}$. We first assume that $\forall t \in I_{t_1}$, $\lambda_t^{BH} \geq \pi_{BH}^* + \varepsilon/2$. Under this assumption, we have following inductive argument: $\forall t \in I_{t_1} - \{t_1\}$, if

$$\frac{\tilde{\lambda}_{T_{\gamma_0}+t}^{BL}}{\tilde{\lambda}_{T_{\gamma_0}+t}^{BH}} \geq \frac{\lambda_{T_{\gamma_0}+t}^{BL}}{\lambda_{T_{\gamma_0}+t}^{BH}}; \ \tilde{\lambda}_{T_{\gamma_0}+t}^{BH} \geq \lambda_{T_{\gamma_0}+t}^{BH}; \tag{52}$$

then

$$\frac{\tilde{\lambda}_{T_{\gamma_0}+t+1}^{BL}}{\tilde{\lambda}_{T_{\gamma_0}+t+1}^{BH}} \geq \frac{\lambda_{T_{\gamma_0}+t+1}^{BL}}{\lambda_{T_{\gamma_0}+t+1}^{BH}}; \ \tilde{\lambda}_{T_{\gamma_0}+t+1}^{BH} \geq \lambda_{T_{\gamma_0}+t+1}^{BH}; \tag{53}$$

The proof for the inductive argument is as following:

$$\frac{\tilde{\lambda}_{T_{\gamma_0}+t+1}^{BL}}{\tilde{\lambda}_{T_{\gamma_0}+t+1}^{BH}} = \frac{\tilde{\lambda}_{T_{\gamma_0}+t}^{BL}}{\tilde{\lambda}_{T_{\gamma_0}+t}^{BH}} \frac{\phi(b|BH, \overline{x}^{up})}{\phi(b|AL, \overline{x}^{up})} \geq \frac{\lambda_{T_{\gamma_0}+t}^{BL}}{\lambda_{T_{\gamma_0}+t}^{BH}} \frac{\phi(b|BH, x_{T_{\gamma_0}+t})}{\phi(b|AL, x_{T_{\gamma_0}+t})}. \tag{54}$$

By assumption, $\lambda_{T_{\gamma_0}+t}^{BH} \geq \pi_{BH}^* + \varepsilon/2$. Also from claim 50, $c_E^{up} > \frac{\tilde{\lambda}_{T_\gamma+t_1}^{BL}}{\tilde{\lambda}_{T_\gamma+t_1}^{BH}} > \frac{\tilde{\lambda}_{T_\gamma+t}^{BL}}{\tilde{\lambda}_{T_\gamma+t}^{BH}}$. Following claim 36, $x_{T_\gamma+t} \in [x_{BH} + \delta^{up}, 1]$. Then the inequality follows this, the inductive assumption $\frac{\tilde{\lambda}_{T_{\gamma_0}+t}^{BL}}{\tilde{\lambda}_{T_{\gamma_0}+t}^{BH}} \geq \frac{\lambda_{T_{\gamma_0}+t}^{BL}}{\lambda_{T_{\gamma_0}+t}^{BH}}$, and the definition that $\overline{x}^{up} = \text{argmax}_{x \in [x_{BH}+\delta^{up},1]} \frac{\phi(b|BL,x)}{\phi(b|BH,x)}$. The proof for $\tilde{\lambda}_{T_{\gamma_0}+t+1}^{BH} \geq \lambda_{T_{\gamma_0}+t+1}^{BH}$ is similar. We also verify that inductive assumption holds for $t = 0$.

Following the inductive proof, we must have $\tilde{\lambda}_{T_{\gamma_0}+t_1}^{BH} \geq \lambda_{T_{\gamma_0}+t_1}^{BH}$. However, in claim 50 we have $\tilde{\lambda}_{T_{\gamma_0}+t_1}^{BH} < \pi_{BH}^* + \varepsilon/2$. This contradicts the assumption that $\lambda_{T_\gamma+t}^{BH} \geq \pi_{BH}^* + \varepsilon/2$ for all $t \in I_{t_1}$.

Now let $t_0 = \min\{t | \lambda_{T_{\gamma_0}+t}^{BH} < \pi_{BH}^* + \varepsilon/2\}$. Then $\lambda_{T_{\gamma_0}+t}^{BH} \geq \pi_{BH}^* + \varepsilon/2$ for all $t \in \{0, 1, \ldots, t_0 - 1\}$. The above inductive argument still works for $t \in \{0, \ldots, t_0 - 2\}$. Therefore, we have that $c_E^{up} > \frac{\tilde{\lambda}_{T_{\gamma_0}+t_1}^{BL}}{\tilde{\lambda}_{T_{\gamma_0}+t_1}^{BH}} > \frac{\lambda_{T_{\gamma_0}+t}^{BL}}{\lambda_{T_{\gamma_0}+t}^{BH}}$ for all $t \in \{0, 1, \ldots, t_0 - 1\}$. By definition of t_0, $\lambda_{T_{\gamma_0}+t}^{BH} \geq \pi_{BH}^* + \varepsilon/2$ for all $t \in \{0, 1, \ldots, t_0 - 1\}$. Use claim 36 again, $x_{T_{\gamma_0}+t} \in [x_{BH} + \delta^{up}, 1]$ for all $t \in \{0, 1, \ldots, t_0 - 1\}$. Since $\frac{\phi(b|BH,x)}{\phi(b|AL,x)} < 1$ for all $x > x_{BH}$, we have $\lambda_{T_{\gamma_0}+t_0-1}^{BH} < \lambda_{T_{\gamma_0}}^{BH} < \overline{\lambda}^{BH}$. Furthermore, $\frac{\lambda_{T_{\gamma_0}+t_0-1}^{BL}}{\lambda_{T_{\gamma_0}+t_0-1}^{BH}}, \frac{\lambda_{T_{\gamma_0}+t_0-1}^{AH}}{\lambda_{T_{\gamma_0}+t_0-1}^{BH}} < c_E^{up}$. Therefore, $\lambda_{T_{\gamma_0}+t_0-1}^{AH}, \lambda_{T_{\gamma_0}+t_0-1}^{BL} < c_E^{up}\overline{\lambda}^{BH}$.

To summarize, up to this point, we have proved that: *For all small enough c_E^{up}, $\exists\ \gamma_0$ and $t_0(\gamma_0, T_{\gamma_0})$ such that*

$$\Lambda_{T_{\gamma_0}+t_0-1} \in [0, c_E^{up}\overline{\lambda}^{BH}]^2 \times [\pi^*_{BH} + \varepsilon/2, \overline{\lambda}^{BH}]. \tag{55}$$

From assumption 49, we have $\overline{\lambda}^{BH} < \frac{\overline{s}}{1-\overline{s}}$. So we can use lemma 26 to find a c_E^{up} small enough such that $\lambda^{BH}_{T_{\gamma_0}+t_0} \geq \pi^*_{BH}$. Therefore, $\lambda^{BH}_{T_{\gamma_0}+t_0} \in [\pi^*_{BH}, \pi^*_{BH} + \varepsilon/2)$. Furthermore, we have $\frac{\lambda^{AH}_{T_{\gamma_0}+t_0}}{\lambda^{BH}_{T_{\gamma_0}+t_0}} < \frac{\lambda^{AH}_{T_{\gamma_0}}}{\lambda^{BH}_{T_{\gamma_0}}} < c_E^{up}$. So $\lambda^{AH}_{T_{\gamma_0}+t_0} \leq \varepsilon/2$ as long as $c_E^{up} < \frac{\varepsilon/2}{\pi^*_{BH}+\varepsilon/2}$. Finally, $\frac{\lambda^{BL}_{T_{\gamma_0}+t_0}}{\lambda^{BH}_{T_{\gamma_0}+t_0}} = \frac{\lambda^{BL}_{T_{\gamma_0}+t_0-1}}{\lambda^{BH}_{T_{\gamma_0}+t_0-1}} \frac{\phi(b|BH, x_{T_{\gamma_0}+t_0-1})}{\phi(b|AL, x_{T_{\gamma_0}+t_0-1})} < c_E^{up} \frac{\phi(b|BH, x_{BH}+\delta^{up})}{\phi(b|AL, x_{BH}+\delta^{up})}$. Here the last inequality following from that $\frac{\phi(b|BH,x)}{\phi(b|AL,x)}$ monotonically decreases on $(x_{BH}, \overline{s})$. (See result 2 in claim 38). Therefore, $\lambda^{BL}_{T_{\gamma_0}+t_0} < \varepsilon/2$ as long as $c_E^{up} < \frac{\varepsilon/2}{\pi^*_{BH}+\varepsilon/2} \frac{\phi(b|BH, x_{BH}+\delta^{up})}{\phi(b|AL, x_{BH}+\delta^{up})}$. ■

Lemma 29 *If $\exists\overline{\lambda}^{BH} < \frac{\overline{s}}{1-\overline{s}}$, and sub-sequence t_k such that*

$$\lambda^{BH}_{t_k}(\mathfrak{h}^{C_1}_{t_k}|\Lambda) < \overline{\lambda}^{BH}.$$

Then

$$\lambda^{AH}_{t_k}(\mathfrak{h}^{C_1}_{t_k}|\Lambda) \to 0; \lambda^{BL}_{t_k}(\mathfrak{h}^{C_1}_{t_k}|\Lambda) \to 0$$

and

$$x_{t_k}(\mathfrak{h}^{C_1}_{t_k}|\Lambda) > x_0 \text{ for sufficiently large } t_k.$$

Proof. Following lemma 16, we must have

$$\frac{\lambda^{AH}(\mathfrak{h}^{C_1}_{t_k}|\Lambda)}{\lambda^{BH}(\mathfrak{h}^{C_1}_{t_k}|\Lambda)} + \frac{\lambda^{BL}(\mathfrak{h}^{C_1}_{t_k}|\Lambda)}{\lambda^{BH}(\mathfrak{h}^{C_1}_{t_k}|\Lambda)} \to 0.$$

If $\lambda^{BH}(\mathfrak{h}^{C_1}_{t_k}|\Lambda) < \overline{\lambda}^{BH}$, we must have

$$\lambda^{AH}_{t_k}(\mathfrak{h}^{C_1}_{t_k}|\Lambda) \to 0; \lambda^{BL}_{t_k}(\mathfrak{h}^{C_1}_{t_k}|\Lambda) \to 0$$

That $x_{t_k}(\mathfrak{h}^{C_1}_{t_k}|\Lambda) > x_0$ for sufficiently large t_k follows directly from claims 17 and 18. ■

Combine previous three lemmas, we have following proposition:

Proposition 30 *If $\exists\overline{\lambda}^{BH} < \frac{\overline{s}}{1-\overline{s}}$, and sub-sequence t_k such that*

$$\lambda^{BH}_{t_k}(\mathfrak{h}^{C_1}_{t_k}|\Lambda) < \overline{\lambda}^{BH}.$$

Then ∃ a finite sequence $\mathfrak{h}_{t_0}^{C}$ such that

$$\|\Lambda(\mathfrak{h}_{t_0}^{C}|\Lambda) - \Lambda^*\| < \varepsilon.$$

Proof. If $\exists \overline{\lambda}^{BH} < \frac{\overline{s}}{1-\overline{s}}$, and sub-sequence t_k such that

$$\lambda_{t_k}^{BH}(\mathfrak{h}_{t_k}^{C_1}|\Lambda) < \overline{\lambda}^{BH}.$$

Then following lemma 29, $\forall \gamma > 0$, $\exists T_\gamma$ such that either (1)$x_{T_\gamma} > x_0; \lambda_{T_\gamma}^{AH} < \gamma, \lambda_{T_\gamma}^{BL} < \gamma, \lambda_{T_\gamma}^{BH} < \pi_{BH}^* - \varepsilon/2$; or (2) $\lambda_{T_\gamma}^{AH} < \gamma, \lambda_{T_\gamma}^{BL} < \gamma, \lambda_{T_\gamma}^{BH} \in [\pi_{BH}^* - \varepsilon/2, \pi_{BH}^* + \varepsilon/2]$; or (3) $\lambda_{T_\gamma}^{AH} < \gamma, \lambda_{T_\gamma}^{BL} < \gamma, \lambda_{T_\gamma}^{BH} \in [\pi_{BH}^* + \varepsilon/2, \overline{\lambda}^{BH}]$. In case (1), we cite lemma 27; in case (2), Λ_{T_γ} is in the $\varepsilon-$neighborhood; in case (3), we cite lemma 28. ■

If there is no such $\overline{\lambda}^{BH} < \frac{\overline{s}}{1-\overline{s}}$, there are two possibilities: either (1) $\lambda^{BH}(\mathfrak{h}_t^{C_1}|\Lambda) \to +\infty$; (2) $\lambda^{BH}(\mathfrak{h}_t^{C_1}|\Lambda)$ doesn't approach $+\infty$, but $\exists \overline{t}$ such that $\lambda_t^{BH}(\mathfrak{h}^{C_1}|\Lambda) \geq \frac{\overline{s}}{1-\overline{s}}$ for all $t \geq \overline{t}$ provided that private signal is bounded. Following lemma 23, in both cases we have a sub-sequence $T_k + (T_k)^2$ and we know the limit action frequency in this sub-sequence. In the next proposition, we make use of this limit frequency to construct an upbound $\overline{\lambda}^{BH} < \frac{\overline{s}}{1-\overline{s}}$, and show that we can push λ^{BH} below the upbound, while keeping λ^{AH} and λ^{BL} arbitrarily small.

Proposition 31 *If (1) $\lambda^{BH}(\mathfrak{h}_t^{C_1}|\Lambda) \to +\infty$; or (2) private signal is bounded, $\lambda^{BH}(\mathfrak{h}_t^{C_1}|\Lambda)$ doesn't approach $+\infty$, but $\exists \overline{t}$ such that $\lambda_t^{BH}(\mathfrak{h}^{C_1}|\Lambda) \geq \frac{\overline{s}}{1-\overline{s}}$ for all $t \geq \overline{t}$. Provided that*

$$\frac{\log \phi(a|AL, \overline{s}) - \log \phi(a|BL, \overline{s})}{\log \phi(b|BL, \overline{s}) - \log \phi(b|AL, \overline{s})} > \frac{\log \phi(a|BH, \overline{s}) - \log \phi(a|AL, \overline{s})}{\log \phi(b|AL, \overline{s}) - \log \phi(b|BH, \overline{s})}; \tag{56}$$

then we can find a finite upper bound $\overline{\lambda}^{BH} < \frac{\overline{s}}{1-\overline{s}}$, and a finite sequence $\mathfrak{h}_{t_0}^{C_2}(\gamma)$ for each small $\gamma > 0$ such that

$$\lambda_t^{AH}(\mathfrak{h}_{t_0}^{C_2}|\Lambda) < \gamma; \lambda_t^{BL}(\mathfrak{h}_{t_0}^{C_2}|\Lambda) < \gamma; \frac{x_0}{1-x_0}(1+\gamma) < \lambda_t^{BH}(\mathfrak{h}_{t_0}^{C_2}|\Lambda) < \overline{\lambda}^{BH}.$$

Let us first sketch the strategy of proof. Condition 56 says, there exists a $\overline{x}$ close to $\overline{s}$, and a positive number r, such that

$$\lambda_{T_k+(T_k)^2}^{BH}\left(\frac{\phi(b|BH, \overline{x})}{\phi(b|AL, \overline{x})}\right)^{\lceil r(T_k)^2 \rceil}, \lambda_{T_k+(T_k)^2}^{BL}\left(\frac{\phi(b|BL, \overline{x})}{\phi(b|AL, \overline{x})}\right)^{\lceil r(T_k)^2 \rceil} \to 0$$

as $T_k \to +\infty$. Define $\overline{\lambda}^{BH} = \frac{1}{\frac{1-\overline{x}}{\overline{x}} - c_E^{up}}$. For c_E^{up} small enough, $\overline{\lambda}^{BH} \approx \frac{\overline{x}}{1-\overline{x}} < \frac{\overline{s}}{1-\overline{s}}$. By the assumption, λ^{BH} eventually moves close to $\frac{\overline{s}}{1-\overline{s}}$. So we could find a big T_k such that $\lambda^{BH}_{T_k+(T_k)^2}$ above $\overline{\lambda}^{BH}$ and $\lambda^{BL}_{T_k+(T_k)^2}\left(\frac{\phi(b|BL,\overline{x})}{\phi(b|AL,\overline{x})}\right)^{\lceil r(T_k)^2 \rceil}$ super small. Then $x_{T_k+(T_k)^2}$ must be above $\overline{x}$. We further verify that $x > \overline{x}$ implies

$$\frac{\phi(b|BH,\overline{x})}{\phi(b|AL,\overline{x})} > \frac{\phi(b|BH,x)}{\phi(b|AL,x)}, \frac{\phi(b|BL,\overline{x})}{\phi(b|AL,\overline{x})} > \frac{\phi(b|BL,x)}{\phi(b|AL,x)}.$$

So

$$\lambda^{BH}_{T_k+(T_k)^2}\left(\frac{\phi(b|BH,\overline{x})}{\phi(b|AL,\overline{x})}\right) > \lambda^{BH}_{T_k+(T_k)^2+1}; \tag{57}$$

and $\lambda^{BL}_{T_k+(T_k)^2+1}$ stays small for that

$$\lambda^{BL}_{T_k+(T_k)^2}\left(\frac{\phi(b|BL,\overline{x})}{\phi(b|AL,\overline{x})}\right)^{\lceil r(T_k)^2 \rceil} > \lambda^{BL}_{T_k+(T_k)^2}\left(\frac{\phi(b|BL,\overline{x})}{\phi(b|AL,\overline{x})}\right) > \lambda^{BL}_{1+T_k+(T_k)^2}. \tag{58}$$

If we also have $\lambda^{BH}_{T_k+(T_k)^2+1} \geq \overline{\lambda}^{BH}$, then by a similar argument we have

$$\lambda^{BH}_{T_k+(T_k)^2}\left(\frac{\phi(b|BH,\overline{x})}{\phi(b|AL,\overline{x})}\right)^2 > \lambda^{BH}_{T_k+(T_k)^2+2}; \tag{59}$$

and that

$$\lambda^{BL}_{T_k+(T_k)^2}\left(\frac{\phi(b|BL,\overline{x})}{\phi(b|AL,\overline{x})}\right)^{\lceil r(T_k)^2 \rceil} > \lambda^{BL}_{T_k+(T_k)^2}\left(\frac{\phi(b|BL,\overline{x})}{\phi(b|AL,\overline{x})}\right)^2 > \lambda^{BL}_{2+T_k+(T_k)^2}. \tag{60}$$

We can proceed inductively and see that λ^{BH} must move below $\overline{\lambda}^{BH}$ at a period t_0 before period $T_k + (T_k)^2 + \lceil r(T_k)^2 \rceil$. Otherwise,

$$\lambda^{BH}_{T_k+(T_k)^2}\left(\frac{\phi(b|BH,\overline{x})}{\phi(b|AL,\overline{x})}\right)^{\lceil r(T_k)^2 \rceil} > \lambda^{BH}_{T_k+(T_k)^2++\lceil r(T_k)^2 \rceil} \geq \overline{\lambda}^{BH}, \tag{61}$$

which contradicts that $\lambda^{BH}_{T_k+(T_k)^2+\lceil r(T_k)^2 \rceil}$ are small for large T_k. Furthermore, up to period t_0, we still have

$$\lambda^{BL}_{T_k+(T_k)^2}\left(\frac{\phi(b|BL,\overline{x})}{\phi(b|AL,\overline{x})}\right)^{\lceil r(T_k)^2 \rceil} > \lambda^{BL}_{T_k+(T_k)^2}\left(\frac{\phi(b|BL,\overline{x})}{\phi(b|AL,\overline{x})}\right)^{t_0} > \lambda^{BL}_{t_0+T_k+(T_k)^2}. \tag{62}$$

So $\lambda^{BL}_{t_0+T_k+(T_k)^2}$ stays super small. Below we have the rigorous proof.

Proof. Let f_a and f_b being as defined in 25. Then condition 56 is equivalent to $\exists r > 0$ s.t.

$$\begin{aligned}
(\frac{\phi(a|BH,\overline{s})}{\phi(a|AL,\overline{s})})^{f_a}(\frac{\phi(b|BH,\overline{s})}{\phi(b|AL,\overline{s})})^{f_b}(\frac{\phi(b|BH,\overline{s})}{\phi(b|AL,\overline{s})})^{r} &< 1; \\
(\frac{\phi(a|BL,\overline{s})}{\phi(a|AL,\overline{s})})^{f_a}(\frac{\phi(b|BL,\overline{s})}{\phi(b|AL,\overline{s})})^{f_b}(\frac{\phi(b|BL,\overline{s})}{\phi(b|AL,\overline{s})})^{r} &< 1.
\end{aligned} \tag{63}$$

Let us pick such a r and fix it through this proof. Since $\phi(b|BH,x), \phi(b|AL,x), \phi(b|BL,x)$ are all continuous in x, we could find a $\overline{x} < \overline{s}$ such that

$$\begin{aligned}
(\frac{\phi(a|BH,\overline{s})}{\phi(a|AL,\overline{s})})^{f_a}(\frac{\phi(b|BH,\overline{s})}{\phi(b|AL,\overline{s})})^{f_b}(\frac{\phi(b|BH,\overline{x})}{\phi(b|AL,\overline{x})})^{r} &< 1; \\
(\frac{\phi(a|BL,\overline{s})}{\phi(a|AL,\overline{s})})^{f_a}(\frac{\phi(b|BL,\overline{s})}{\phi(b|AL,\overline{s})})^{f_b}(\frac{\phi(b|BL,\overline{x})}{\phi(b|AL,\overline{x})})^{r} &< 1.
\end{aligned} \tag{64}$$

We also pick and fix a $\overline{x}$ throughout this proof.

As argued in lemma 23, in both cases (1) and (2) , $\forall k \in \mathbb{N}$, $\exists T_k \in \mathbb{N}$, s.t. $x_t \in (\overline{s} - \frac{1}{k}, 1]$ for all $t \geq T_k$. In particular, let us choose T_k as constructed in lemma 23. For each k and T_k, we can construct an associated auxiliary process as following: Let $\tilde{\Lambda}_{T_k+(T_k)^2} = \Lambda_{T_k+(T_k)^2}$, for each $t \in \{T_k + (T_k)^2 + 1, \ldots, T_k + (T_k)^2 + \lceil r(T_k)^2 \rceil\}$, define $\tilde{\Lambda}_t$'s evolution as following

$$\begin{aligned}
\tilde{\lambda}^{AH}_{t+1} &= \tilde{\lambda}^{AH}_t \frac{\phi(b|AH,\overline{x})}{\phi(b|AL,\overline{x})}; \\
\tilde{\lambda}^{BL}_{t+1} &= \tilde{\lambda}^{BL}_t \frac{\phi(b|BL,\overline{x})}{\phi(b|AL,\overline{x})}; \\
\tilde{\lambda}^{BH}_{t+1} &= \tilde{\lambda}^{BH}_t \frac{\phi(b|BH,\overline{x})}{\phi(b|AL,\overline{x})}.
\end{aligned}$$

The idea for this construction is to use $\tilde{\lambda}^{BL}$ to control how fast λ^{BL} can increase; and use $\tilde{\lambda}^{BH}$ to control how fast λ^{BH} can decrease.

For each k and T_k, we have that

$$\begin{aligned}
& \tilde{\lambda}^{BL}_{T_k+(T_k)^2+\lceil r(T_k)^2\rceil}\frac{\phi(b|AL,\overline{x})}{\phi(b|BL,\overline{x})} \leq \lambda^{BL}_{T_k+(T_k)^2}\left(\frac{\phi(b|BL,\overline{x})}{\phi(b|AL,\overline{x})}\right)^{r(T_k)^2} \\
= & \lambda^{BL}_{T_k}\left(\Pi_{t=T_k}^{T_k+(T_k)^2-1}\frac{\phi(\mathfrak{h}_t^{C_1}|BL,x_t)}{\phi(\mathfrak{h}_t^{C_1}|AL,x_t)}\right)\left(\frac{\phi(b|BL,\overline{x})}{\phi(b|AL,\overline{x})}\right)^{r(T_k)^2} \\
\leq & \lambda^{BL}_{T_k}\left\{\left(\frac{\phi(b|BL,s_k^{*b})}{\phi(b|AL,s_k^{*b})}\right)^{\frac{\#b}{(T_k)^2}}\left(\frac{\phi(a|BL,s_k^{*a})}{\phi(a|AL,s_k^{*a})}\right)^{\frac{\#a}{(T_k)^2}}\left(\frac{\phi(b|BL,\overline{x})}{\phi(b|AL,\overline{x})}\right)^{r}\right\}^{(T_k)^2}, \qquad (65)
\end{aligned}$$

where $s_k^{*\alpha} \equiv \text{argmax}_{x\in[\overline{s}-\frac{1}{k},1]}\frac{\phi(b|BL,x)}{\phi(b|AL,x)}$, $\alpha \in \{a,b\}$. Here the first inequality just takes care of the case that $r(T_k)^2$ is not an integer. The second inequality follows from that $x_t \in (\overline{s}-\frac{1}{k},1]$, when $t > T_k$. Recall that $\frac{\phi(\alpha|BL,x)}{\phi(\alpha|AL,x)} = \frac{\phi(\alpha|BL,\overline{s})}{\phi(\alpha|AL,\overline{s})}$ for $x \in [\overline{s},1]$. So we have, for sufficiently large k and T_k, the big term within the curly bracket in 65 is sufficiently close to $\left(\frac{\phi(a|BL,\overline{})}{\phi(a|AL,\overline{s})}\right)^{f_a}\left(\frac{\phi(b|BL,\overline{s})}{\phi(b|AL,\overline{s})}\right)^{f_b}\left(\frac{\phi(b|BL,\overline{x})}{\phi(b|AL,\overline{x})}\right)^{r}$, which is strictly below 1 (see 64).

Similarly, for each k and T_k, we have that

$$\begin{aligned}
& \tilde{\lambda}^{BH}_{T_k+(T_k)^2+\lceil r(T_k)^2\rceil} \\
\leq & \lambda^{BH}_{T_k}\left\{\left(\frac{\phi(b|BH,s_k^{*b})}{\phi(b|AL,s_k^{*b})}\right)^{\frac{\#b}{(T_k)^2}}\left(\frac{\phi(a|BH,s_k^{*a})}{\phi(a|AL,s_k^{*a})}\right)^{\frac{\#a}{(T_k)^2}}\left(\frac{\phi(b|BH,\overline{x})}{\phi(b|AL,\overline{x})}\right)^{r}\right\}^{(T_k)^2}, \qquad (66)
\end{aligned}$$

where $s_k^{*\alpha} \equiv \text{argmax}_{x\in[\overline{s}-\frac{1}{k},1]}\frac{\phi(b|BH,x)}{\phi(b|AL,x)}$, $\alpha \in \{a,b\}$. (Here we use the same notation as in 65 just to avoid too many notations.) For sufficiently large k and T_k, the big term in 66 is sufficiently close to $\left(\frac{\phi(a|BH,\overline{s})}{\phi(a|AL,\overline{s})}\right)^{f_a}\left(\frac{\phi(b|BH,\overline{s})}{\phi(b|AL,\overline{s})}\right)^{f_b}\left(\frac{\phi(b|BH,\overline{x})}{\phi(b|AL,\overline{x})}\right)^{r}$, which is strictly below 1.

Now choose and fix a proper k, we have

$$\lim_{T_k\to+\infty}\tilde{\lambda}^{BH}_{T_k+(T_k)^2+\lceil r(T_k)^2\rceil} = 0; \lim_{T_k\to+\infty}\tilde{\lambda}^{BL}_{T_k+(T_k)^2+\lceil r(T_k)^2\rceil} = 0.$$

Arbitrarily choose and fix a $\gamma > 0$. For all

$$0 < c_E^{up} < \min\{\frac{\varepsilon/2}{\pi^*_{BH}+\varepsilon/2}, \frac{\varepsilon/2}{\pi^*_{BH}(\pi^*_{BH}+\varepsilon/2)}, \frac{\gamma}{\gamma+\frac{\phi(b|AL,1)}{\phi(b|BH,1)}}\frac{1-\overline{x}}{\overline{x}}, \frac{1-\overline{x}}{\overline{x}} - \frac{1-\overline{s}}{\overline{s}}\},$$

define $\overline{\lambda}^{BH} = \frac{1}{\frac{1}{\overline{x}}-1-c_E^{up}}$. For any γ, let us choose a T_k such that

$$\begin{aligned}
&\tilde{\lambda}^{BL}_{T_k+(T_k)^2+\lceil r(T_k)^2\rceil} < \gamma;\\
&\tilde{\lambda}^{BH}_{T_k+(T_k)^2+\lceil r(T_k)^2\rceil} < \overline{\lambda}_{BH};\\
&\frac{\lambda^{AH}_{T_k+(T_k)^2}}{\lambda^{BH}_{T_k+(T_k)^2}} < c_E^{up};\\
&\lambda^{BH}_{T_k+(T_k)^2} > \overline{\lambda}^{BH}.
\end{aligned} \tag{67}$$

Here because $c_E^{up} < \frac{1-\overline{x}}{\overline{x}} - \frac{1-\overline{s}}{\overline{s}}$, we have $\overline{\lambda}^{BH} < \frac{\overline{s}}{1-\overline{s}}$. We also have $\overline{\lambda}^{BH} > \pi^*_{BH}$ always holds. Moreover, $c_E^{up} < \frac{\gamma}{\gamma + \frac{\phi(b|AL,1)}{\phi(b|BH,1)}} \frac{1-\overline{x}}{\overline{x}}$ implies that $c_E^{up} < \frac{\gamma}{\overline{\lambda}^{BH} \frac{\phi(b|AL,1)}{\phi(b|BH,1)}}$.

We claim: for the choosen T_k, there exists a $t \in \{T_k + (T_k)^2, \ldots, T_k + (T_k)^2 + \lceil r(T_k)^2\rceil\}$ (abbreviate this index set as I_{T_k} from now on) such that $\lambda_t^{BH} < \overline{\lambda}^{BH}$. Assume not, then $\forall t \in I_{T_k}$, $\lambda_t^{BH} \geq \overline{\lambda}^{BH}$. Besides, $\forall t \in I_{T_k}$, we have $\frac{\lambda_t^{AH}}{\lambda_t^{BH}} < \frac{\lambda^{AH}_{T_k+(T_k)^2}}{\lambda^{BH}_{T_k+(T_k)^2}} < c_E^{up}$ since $\frac{\phi(b|AH,x)}{\phi(b|BH,x)} < 1$ always holds. We must have $x_t \geq \overline{x}$ for all $t \in I_{T_k}$ following a similar argument as in 69.

Because $\overline{x} = \mathrm{argmax}_{x\in[\overline{x},1]} \frac{\phi(b|BL,x)}{\phi(b|AL,x)} = \mathrm{argmax}_{x\in[\overline{x},1]} \frac{\phi(b|BH,x)}{\phi(b|AL,x)}$. (See claim 38), we could build up following inductive argument for all $t \in I_{T_k}$: that $\tilde{\lambda}_t^{BH} \geq \lambda_t^{BH}$ and $\tilde{\lambda}_t^{BL} \geq \lambda_t^{BL}$ implies $\tilde{\lambda}_{t+1}^{BH} \geq \lambda_t^{BH}$ and $\tilde{\lambda}_{t+1}^{BL} \geq \lambda_t^{BL}$. The proof is direct: $\tilde{\lambda}_{t+1}^{BH} = \tilde{\lambda}_t^{BH} \frac{\phi(b|BH,\overline{x})}{\phi(b|AL,\overline{x})} \geq \lambda_t^{BH} \frac{\phi(b|BH,\overline{x})}{\phi(b|AL,\overline{x})} \geq \lambda_t^{BH} \frac{\phi(b|BH,x_t)}{\phi(b|AL,x_t)} = \lambda_{t+1}^{BH}$. The first inequality follows from the inductive hypothesis, the second inequality follows from that $x_t \geq \overline{x}$ for all $x \in I_{T_k}$.

This inductive argument leads to a contradiction:

$$\overline{\lambda}^{BH} \leq \lambda^{BH}_{T_k+(T_k)^2+\lceil r(T_k)^2\rceil} \leq \tilde{\lambda}^{BH}_{T_k+(T_k)^2+\lceil r(T_k)^2\rceil} < \overline{\lambda}^{BH}.$$

So there must exists a $t \in I_{T_k}$ such that $\lambda_t^{BH} < \overline{\lambda}^{BH}$. Let $t_0 = \min\{t \in I_{T_k} | \lambda_t^{BH} < \overline{\lambda}^{BH}\}$. Above inductive argument still works for $t \leq t_0 - 1$; so we can conclude that $\lambda_t^{BL} \leq \tilde{\lambda}_t^{BL} < \tilde{\lambda}^{BL}_{T_k+(T_k)^2+\lceil r(T_k)^2\rceil} < \gamma$ for $t \in \{T_k + (T_k)^2, \ldots, t_0\}$. Furthermore, $\frac{\lambda^{AH}_{t_0-1}}{\lambda^{BH}_{t_0-1}} < \frac{\lambda^{AH}_{T_k+(T_k)^2}}{\lambda^{BH}_{T_k+(T_k)^2}} < c_E^{up}$. Also $\lambda^{BH}_{t_0-1} \frac{\phi(b|BH,1)}{\phi(b|AL,1)} < \lambda^{BH}_{t_0-1} \frac{\phi(b|BH,x_{t_0-1})}{\phi(b|AL,x_{t_0-1})} = \lambda^{BH}_{t_0} < \overline{\lambda}^{BH}$, so $\lambda^{BH}_{t_0-1} < \frac{\overline{\lambda}^{BH}}{\frac{\phi(b|BH,1)}{\phi(b|AL,1)}}$. Thus $\lambda^{AH}_{t_0-1} < \gamma$.

Knowing that $\lambda^{AH}_{t_0-1} < \gamma, \lambda^{BL}_{t_0-1} < \gamma$, we have $x_{t_0-1} < \frac{\lambda^{BH}_{t_0-1}+\gamma}{\lambda^{BH}_{t_0-1}+\gamma+1}$. Therefore

$$\begin{aligned}
\lambda^{BH}_{t_0} &= \lambda^{BH}_{t_0-1}\frac{\phi(b|BH, x_{t_0-1})}{\phi(b|AL, x_{t_0-1})} \\
&> \lambda^{BH}_{t_0-1}\frac{\phi(b|BH, \frac{\lambda^{BH}_{t_0-1}+\gamma}{\lambda^{BH}_{t_0-1}+\gamma+1})}{\phi(b|AL, \frac{\lambda^{BH}_{t_0-1}+\gamma}{\lambda^{BH}_{t_0-1}+\gamma+1})} \\
&= (\lambda^{BH}_{t_0-1}+\gamma)\frac{\phi(b|BH, \frac{\lambda^{BH}_{t_0-1}+\gamma}{\lambda^{BH}_{t_0-1}+\gamma+1})}{\phi(b|AL, \frac{\lambda^{BH}_{t_0-1}+\gamma}{\lambda^{BH}_{t_0-1}+\gamma+1})} - \gamma\frac{\phi(b|BH, \frac{\lambda^{BH}_{t_0-1}+\gamma}{\lambda^{BH}_{t_0-1}+\gamma+1})}{\phi(b|AL, \frac{\lambda^{BH}_{t_0-1}+\gamma}{\lambda^{BH}_{t_0-1}+\gamma+1})} \\
&> \pi^*_{BH} - \gamma\frac{\phi(b|BH, \frac{\lambda^{BH}_{t_0-1}+\gamma}{\lambda^{BH}_{t_0-1}+\gamma+1})}{\phi(b|AL, \frac{\lambda^{BH}_{t_0-1}+\gamma}{\lambda^{BH}_{t_0-1}+\gamma+1})} \qquad (68)
\end{aligned}$$

Here the first inequality follows that $\frac{\phi(b|BH,x)}{\phi(b|AL,x)}$ monotonically decreasing. The last inequality follows assumption 25. By choosing γ small enough $\pi^*_{BH} - \gamma\frac{\phi(b|BH, \frac{\lambda^{BH}_{t_0-1}+\gamma}{\lambda^{BH}_{t_0-1}+\gamma+1})}{\phi(b|AL, \frac{\lambda^{BH}_{t_0-1}+\gamma}{\lambda^{BH}_{t_0-1}+\gamma+1})} > \frac{x_0}{1-x_0}(1+\gamma)$.

Finally, We can verify that $\lambda^{AH}_{t_0} < \frac{\gamma}{\frac{\phi(b|AL,1)}{\phi(b|BH,1)}} < \gamma$.

Therefore, for any small $\gamma > 0$, there is a finite $\overline{\lambda}^{BH} < \frac{\bar{s}}{1-\bar{s}}$ and a finite sequence of actions $\mathfrak{h}^{C_2}_{t_0}$ such that

$$\lambda^{AH}_t(\mathfrak{h}^{C_2}_{t_0}|\Lambda) < \gamma; \lambda^{BL}_t(\mathfrak{h}^{C_2}_{t_0}|\Lambda) < \gamma; \frac{x_0}{1-x_0}(1+\gamma) < \lambda^{BH}_t(\mathfrak{h}^{C_2}_{t_0}|\Lambda) < \overline{\lambda}^{BH}.$$

This sequence starts with $\mathfrak{h}^{C_1}_{T_k+(T_k)^2}$ for some large k and large T_k; and ends with a long sequence of action b. ■

It is direct to verify that

$$\lambda^{AH}_t(\mathfrak{h}^{C_2}_{t_0}|\Lambda) < \gamma; \lambda^{BL}_t(\mathfrak{h}^{C_2}_{t_0}|\Lambda) < \gamma; \frac{x_0}{1-x_0}(1+\gamma) < \lambda^{BH}_t(\mathfrak{h}^{C_2}_{t_0}|\Lambda)$$

implies that $x_t(\mathfrak{h}^{C_2}_{t_0}|\Lambda) > x_0$. Therefore, we can again cite lemma 27 and 28 to conclude that, with another finite sequence of action b following $\mathfrak{h}^{C_2}_{t_0}$, society's belief is pushed into the $\varepsilon-$neighborhood.

Following are a few computation results which is used in previous proof. The first claim computes the minimum posterior weight associated to payoff state B, given that current weight is $x \in (x_0, 1]$.

Claim 32 *Consider set* $\Lambda = \{(p_{AH}, p_{BL}, p_{BH}) | x = p_{BH} + p_{BL}, x \in (x_0, 1]\}$, *let*

$$x(b) = \frac{p_{BH}\phi(b|BH,x) + p_{BL}\phi(b|BL,x)}{p_{AH}\phi(b|AH,x) + p_{AL}\phi(b|AL,x) + p_{BH}\phi(b|BH,x) + p_{BL}\phi(b|BL,x)},$$

where $(p_{AH}, p_{BL}, p_{BH}) \in \Lambda$. *Then*

$$x(b) \geq \frac{x\phi(b|BH,x)}{(1-x)\phi(b|AL,x) + x\phi(b|BH,x)}.$$

Proof. Since $p_{BH} + p_{BL} = x$, and $p_{AH} + p_{AL} = 1 - x$, we can rewrite $x(b)$ just in terms of p_{AH} and p_{BH}, where $p_{AH} \in [0, 1-x]$ and $p_{BH} \in [0, x]$. Then we compute and find that $\frac{dx(b)}{dp_{AH}} > 0$, for the reason that $\phi(b|AH,x) - \phi(b|AL,x) < 0$ on $x \in (x_0, 1]$. So

$$x(b) \geq x(b)|_{p_{AH}=0} = \frac{x\phi(b|BL,x) + p_{BH}[\phi(b|BH,x) - \phi(b|BL,x)]}{(1-x)\phi(b|AL,x) + x\phi(b|BL,x) + p_{BH}[\phi(b|BH,x) - \phi(b|BL,x)]}.$$

Similarly we compute and find that $\frac{d}{dp_{BH}}x(b)|_{p_{AH}=0} < 0$, for that $\phi(b|BH,x) - \phi(b|BL,x) < 0$ on $x \in (x_0, 1]$. So

$$x(b)|_{p_{AH}=0} \geq x(b)|_{p_{AH}=0, p_{BH}=x} \geq \frac{x\phi(b|BH,x)}{x\phi(b|BH,x) + (1-x)\phi(b|AL,x)}.$$

■

Claim 33 *Let* $\mathfrak{F}(x) = \frac{\phi(b|BL,x) - \phi(b|BH,x)}{\phi(b|BH,x) - \phi(b|AH,x)}$, *then if*

1. *private signal is unbounded, then* $\mathfrak{F}'(x) > 0$ *on* $x \in (0, 1)$.
2. *private signal is bounded, then* $\mathfrak{F}'(x) > 0$ *on* $x \in (\underline{s}, \overline{s})$; *and* $\mathfrak{F}'(x) = 0$ *on* $x \in (0, \underline{s}] \cup [\overline{s}, 1)$.

Proof. First we compute $\mathfrak{F}'(x)$ Since $f^B(x) = \frac{1-x}{x} f^A(x)$, we can write

$$\mathfrak{F}'(x) = \frac{f^A(x)(p_H - p_L)}{[\phi(b|BH,x) - \phi(b|AH,x)]^2} A(x),$$

where $A(x) = \frac{1-x}{x}[F^B(x_0) - p_H F^A(x_0)] - \frac{1-x}{x}(1-p_H)F^A(x) + (1-p_H)[F^B(x) - F^B(x_0)]$.

We first show that $A(x) > 0$ on $x \in [x_0, 1]$. We can verify that $A(x_0) > 0$ and $A(1) > 0$. Furthermore, we compute $A'(x) = \frac{1}{x^2}[(1-p_H)F^A(x) + p_H F^A(x_0) - F^B(x_0)]$. We can see that either (1) $A'(x) < 0$ on $x \in [x_0, 1]$ or (2) $\exists$ an unique $x^* \in (x_0, 1]$ such that $A'(x^*) = 0$. In the first case, obviously $A(x) > 0$ on $x \in [x_0, 1]$. In the second case, We can see that $A(x)$ achieves minimum $(1 - p_H)[F^B(x^*) - F^B(x_0)] > 0$ at x^*.

Furthermore, we observe that $\lim_{x \to 0^+} A(x) \to +\infty$, and $A'(x) < 0$ on $x \in (0, x_0]$. So $A(x) > 0$ on $(0, x_0]$ as well.

If private signal is unbounded, then $f^A(x) > 0$ on $x \in (0, 1)$. Thus $\mathfrak{F}'(x) > 0$ on $(0, 1)$. If private signal is bounded, then $f^A(x) > 0$ on $x \in (\underline{s}, \overline{s})$; and $f^A(x) = 0$ on $x \in (0, \underline{s}] \cup [\overline{s}, 1)$. And the second conclusion follows directly. ■

Claim 34 $\frac{\phi(b|AH,x)}{\phi(b|BH,x)} \leq 1 + \frac{p_H[F^A(x_0) - F^B(x_0)]}{p_H F^B(x_0) + (1-p_H)} < 1$; $\frac{\phi(a|AH,x)}{\phi(a|BH,x)} \geq 1 + \frac{p_H[F^B(x_0) - F^A(x_0)]}{p_H[1 - F^B(x_0)] + (1-p_H)} > 1$.

Proof.

$$\begin{aligned}
\frac{\phi(b|AH,x)}{\phi(b|BH,x)} - 1 &= \frac{p_H[F^A(x_0) - F^B(x_0)] + (1-p_H)[F^A(x) - F^B(x)]}{p_H F^B(x_0) + (1-p_H)F^B(x)} \\
&\leq \frac{p_H[F^A(x_0) - F^B(x_0)]}{p_H F^B(x_0) + (1-p_H)F^B(x)} \\
&\leq \frac{p_H[F^A(x_0) - F^B(x_0)]}{p_H F^B(x_0) + (1-p_H)}.
\end{aligned}$$

The other inequality can be similarly verified. ■

Claim 35 *If* $x \in [x_0, 1]$, *then*

$$\begin{aligned}
\frac{\phi(b|AH,x)}{\phi(b|BL,x)} &\leq 1 - \frac{\max\{\phi(b|AH,x_0) - \phi(b|BL,x_0),\ \phi(b|AH,1) - \phi(b|BL,1)\}}{p_L F^B(x_0) + (1-p_L)} < 1 \\
\frac{\phi(a|AH,x)}{\phi(a|BL,x)} &\geq 1 - \frac{\max\{\phi(a|AH,x_0) - \phi(a|BL,x_0),\ \phi(a|AH,1) - \phi(a|BL,1)\}}{p_L[1 - F^B(x_0)] + (1-p_L)} > 1
\end{aligned}$$

Proof. Let $f(x) = \phi(b|AH,x) - \phi(b|BL,x)$. It is direct to verify that $f(x_0) < 0$ and $f(1) < 0$. Furthermore, $f'(x) = f^A(x)[(1-p_H) - (1-p_L)\frac{1-x}{x}]$. If private signal is unbounded, then $(1-p_H) - (1-p_L)\frac{1-x}{x}$ strictly increases from $(1-p_H) - (1-p_L)\frac{1-x_0}{x_0}$ to $1 - p_H$. Depends on whether $(1-p_H) - (1-p_L)\frac{1-x_0}{x_0}$ is negative, $f(x)$ either strictly increases or reaches an unique minimum somewhere between x_0 and 1. If private signal is bounded, then $(1-p_H) - (1-p_L)\frac{1-x}{x}$ strictly increases from $(1-p_H) - (1-p_L)\frac{1-x_0}{x_0}$ to $(1-p_H) - (1-p_L)\frac{1-\overline{s}}{\overline{s}}$. If $(1-p_H) - (1-p_L)\frac{1-\overline{s}}{\overline{s}} \leq 0$, then $f(x)$ strictly decreases on $[x_0, 1]$. If $(1-p_H) - (1-p_L)\frac{1-\overline{s}}{\overline{s}} > 0$, then $f(x)$ either strictly increases or reaches an unique minimum somewhere between x_0 and 1.

Therefore, $f(x) \leq \max\{\phi(b|AH, x_0) - \phi(b|BL, x_0),\ \phi(b|AH, 1) - \phi(b|BL, 1)\}$. The first inequality follows directly. The verification of the second inequality is very similar, for the reason that $\phi(a|AH, x) - \phi(a|BL, x) = -f(x)$. ■

Claim 36 *If $\lambda_t^{BH} \geq \pi_{BH} + \frac{\varepsilon}{2}$ and $\frac{\lambda_t^{AH}}{\lambda_t^{BH}} < c_E^{up} < \min\{\frac{\varepsilon/2}{\pi_{BH}(\pi_{BH}+\varepsilon/2)}, \frac{\varepsilon/2}{\pi_{BH}+\varepsilon/2}\}$, then $x_t > \frac{1}{1+c_E^{up}+\frac{1}{\pi_{BH}+\varepsilon/2}} > x_{BH}$. For notation convenience, we denote $\frac{1}{1+c_E^{up}+\frac{1}{\pi_{BH}+\varepsilon/2}}$ as $x_{BH} + \delta^{up}$.*

Proof. We have

$$
\begin{aligned}
x_t &= \frac{\lambda_t^{BH} + \lambda_t^{BL}}{1 + \lambda_t^{AH} + \lambda_t^{BL} + \lambda_t^{BH}} = \frac{1 + \frac{\lambda_t^{BL}}{\lambda_t^{BH}}}{\frac{1}{\lambda_t^{BH}} + \frac{\lambda_t^{AH}}{\lambda_t^{BH}} + \frac{\lambda_t^{BL}}{\lambda_t^{BH}} + 1} \\
&> \frac{1}{\frac{1}{\lambda_t^{BH}} + \frac{\lambda_t^{AH}}{\lambda_t^{BH}} + 1} > \frac{1}{1 + c_E^{up} + \frac{1}{\pi_{BH}+\varepsilon/2}}.
\end{aligned} \tag{69}
$$

It is direct to verify that $c_E^{up} < \frac{\varepsilon/2}{\pi_{BH}(\pi_{BH}+\varepsilon/2)}$ is equivalent to that $\frac{1}{1+c_E^{up}+\frac{1}{\pi_{BH}+\varepsilon/2}} > \frac{\pi_{BH}}{\pi_{BH}+1}$. ■

Claim 37 *If $\lambda_t^{BH} \leq \pi_{BH} - \varepsilon/2$ and $\frac{\lambda_t^{BL}}{\lambda_t^{BH}} < c_E^{down} < \frac{\varepsilon/2}{\pi_{BH}-\varepsilon/2}$. Then $x_t < \frac{1+c_E^{down}}{1+c_E^{down}+\frac{1}{\pi_{BH}-\varepsilon/2}} < x_{BH}$. For notation convenience, we denote $\frac{1+c_E^{down}}{1+c_E^{down}+\frac{1}{\pi_{BH}-\varepsilon/2}}$ as $x_{BH} - \delta^{down}$.*

Proof. We have

$$
\begin{aligned}
x_t &= \frac{\frac{\lambda_t^{BL}}{\lambda_t^{BH}} + 1}{\frac{1}{\lambda_t^{BH}} + \frac{\lambda_t^{BL}}{\lambda_t^{BH}} + \frac{\lambda_t^{AH}}{\lambda_t^{BH}} + 1} < \frac{\frac{\lambda_t^{BL}}{\lambda_t^{BH}} + 1}{\frac{1}{\lambda_t^{BH}} + \frac{\lambda_t^{BL}}{\lambda_t^{BH}} + 1} \\
&< \frac{1 + c_E^{down}}{1 + c_E^{down} + \frac{1}{\pi_{BH}-\varepsilon/2}}.
\end{aligned}
$$

It is direct to verify that $c_E^{down} < \frac{\varepsilon/2}{\pi_{BH}-\varepsilon/2}$ is equivalent to that $\frac{1+c_E^{down}}{1+c_E^{down}+\frac{1}{\pi_{BH}-\varepsilon/2}} < \frac{\pi_{BH}}{\pi_{BH}+1}$. ■

Claim 38 *We have following results:*

1. *$\frac{\phi(b|BL,x)}{\phi(b|BH,x)}$ is strictly increasing on $(\underline{s}, \overline{s})$, and is constant on $(0, \underline{s}) \cup (\overline{s}, 1)$.*
2. *$\frac{\phi(b|AL,x)}{\phi(b|BH,x)}$ is strictly increasing on $(x_0, \overline{s})$, and is constant on $(\overline{s}, 1)$.*
3. *$\frac{\phi(b|BL,x)}{\phi(b|AL,x)}$ is strictly decreasing on $(x_{BH}, \overline{s})$, and is constant on $(\overline{s}, 1)$.*
4. *$\frac{\phi(a|BL,x)}{\phi(a|AL,x)}$ is weakly increasing on $x \in (1-\varepsilon, 1)$ for any small enough ε.*

Proof. To see the first result, we compute $\frac{d}{dx}\frac{\phi(b|BL,x)}{\phi(b|BH,x)} = f^B(x)F^B(x_0)(p_H - p_L)$, which is strictly positive on $(\underline{s}, \overline{s})$ and 0 on $(0, \underline{s}) \cup (\overline{s}, 1)$.

To see the second result, we compute that $\frac{d}{dx}\frac{\phi(b|AL,x)}{\phi(b|BH,x)} = \frac{f^A(x)}{[\phi(b|BH,x)]^2}g(x)$, where $g(x) = \left[(1-p_L)\phi(b|BH,x) - (1-p_H)\phi(b|AL,x)\frac{1-x}{x}\right]$. We can prove that $g(x) > 0$ on $x \in (x_0, \overline{s})$ as following: first, as $x \to x_0$, we have $g(x) \to (1-p_L)F^B(x_0) - (1-p_H)F^A(x_0)\frac{1-x_0}{x_0}$, which is strictly positive since $F^B(x_0) = \int_0^{x_0} \frac{1-t}{t} dF^A(t) \geq \frac{1-x_0}{x_0}F^A(x_0)$; second, we compute $g'(x) = (1-p_H)\phi(b|AL,x)\frac{1}{x^2} > 0$ on $x \in (0,1)$.

To see the third result, we similarly compute $\frac{d}{dx}\frac{\phi(b|BL,x)}{\phi(b|AL,x)} = \frac{(1-p_L)f^A(x)}{[\phi(b|AL,x)]^2}h(x)$, where $h(x) = \frac{1-x}{x}\phi(b|AL,x) - \phi(b|BL,x)$. We can prove that $h(x) < 0$ on $x \in (x_{BH}, 1)$ as following: first, we compute $h'(x) = -\frac{1}{x^2}\phi(b|AL,x) < 0$ on $x \in (x_{BH}, 1)$; second, we can prove that as $x \to x_{BH}$, $g(x) \to \frac{1-x_{BH}}{x_{BH}}\phi(b|AL,x_{BH}) - \phi(b|BL,x_{BH}) < 0$. Here, we need to use the fact that $F^B(x) = \int_0^x \frac{1-t}{t} dF^A(x) \geq \frac{1-x}{x}F^A(x)$ for all $x \in (0, \overline{s})$. Then $\frac{1-x_{BH}}{x_{BH}}\phi(b|AL,x) - \phi(b|BL,x_{BH}) = p_L[\frac{1-x_{BH}}{x_{BH}}F^A(x_0) - F^B(x_0)] + (1-p_L)[\frac{1-x_{BH}}{x_{BH}}F^A(x_{BH}) - F^B(x_{BH})]$, where $\frac{1-x_{BH}}{x_{BH}}F^A(x_0) - F^B(x_0) < \frac{1-x_0}{x_0}F^A(x_0) - F^B(x_0) \leq 0$; and $\frac{1-x_{BH}}{x_{BH}}F^A(x_{BH} - F^B(x_{BH})) \leq 0$.

To see the fourth result, we compute $\frac{d}{dx}\frac{\phi(a|BL,x)}{\phi(a|AL,x)} = (1-p_L)f^A(x)[-\frac{1-x}{x}\phi(a|AL,x) + \phi(a|BL,x)]$. If private signal is of bounded strength, then obvious this derivative is 0; if private signal is of unbounded strength, then we can always find a small enough ε to guarantee that $-\frac{1-x}{x}\phi(a|AL,x) + \phi(a|BL,x) > 0$. ■

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