Macroeconomics under Imperfect Knowledge

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Motivation

- Macroeconomic models depend on household and firm expectations
- Policy design
  - Not enough to observe some history of beliefs
  - Need a theory of determination of beliefs: rational expectations
- Rational expectations policy design
  - Heavy reliance on managing expectations through annouced commitments
  - What are the consequences of imprecise control of beliefs?

Motivation

- Reasonable to suppose agents have limited information about policy regime
  - US financial crisis witnessed dramatic change in the nature and conduct of stabilization policy
  - Does imperfect knowledge limit the efficacy of policy?
- Bernanke (2004)

  "[...] most private-sector borrowing and investment decisions depend not on the funds rate but on longer-term yields, such as mortgage rates and corporate bond rates, and on the prices of long-lived assets, such as housing and equities. Moreover, the link between these longer-term yields and asset prices and the current setting of the federal funds rate can be quite loose at times."

Theories of Expectations Formation with Dynamic Models

- Cobweb model:

$$\begin{aligned} Q_t^d &= a - bP_t \\ Q_t^s &= c + dE_{t-1}P_t + u_t \\ Q_t^s &= Q_t^d = Q_t \end{aligned}$$

Demand, supply and equilibrium conditions are given above, respectively

- Supply of the goods in period t depends on the firms' expectations in period $t-1$, u_t is the stochastic disturbance
- Market clears in equilibrium

$$\begin{aligned} a - bP_t &= c + dE_{t-1}P_t + u_t \\ P_t &= \frac{a-c}{b} - \frac{d}{b}E_{t-1}P_t - \frac{1}{b}u_t \end{aligned}$$

Expectations

- Myopic expectations:

$$E_tP_{t+1} = P_t$$

- Adaptive expectations:

$$E_tP_{t+1} = \lambda P_t + (1-\lambda)\, E_{t-1}P_t = E_{t-1}P_t + \lambda\left(P_t - E_{t-1}P_t\right)$$

- Rational Expectations:

$$E_tP_{t+1} = \text{mathematical expectations taken with respect to the probability distribution implied by the model}$$

Myopic Expectations:

- This gives

$$P_t = \frac{a-c}{b} - \frac{d}{b}P_{t-1} - \frac{1}{b}u_t$$

Stable iff

$$\left|\frac{d}{b}\right| < 1$$

- Conditional mathematical expectation

$$E^M_{t-1}P_t = \frac{a-c}{b} - \frac{d}{b}P_{t-1} \neq P_{t-1}$$

Information about the economy available to suppliers when setting their prices is not being well utilized

Adaptive Expectations:

- Nerlove (1958):

$$
\begin{aligned}
E_t P_{t+1} &= \lambda P_t + (1-\lambda) E_{t-1} P_t \\
&= \lambda \sum_{i=0}^{\infty} (1-\lambda)^i P_{t-i}
\end{aligned}
$$

  Price expectations adjust slowly in response to new information revealed in recent prices.

- Adaptive expectations: lag one period $\rightarrow$

$$
\begin{aligned}
P_{t-1} &= \frac{a-c}{b} - \frac{d}{b} E_{t-2} P_{t-1} - \frac{1}{b} u_{t-1} \\
\Rightarrow & \quad E_{t-2} P_{t-1} = \frac{b}{d} \left(\frac{a-c}{b} - P_{t-1} - \frac{1}{b} u_{t-1} \right)
\end{aligned}
$$

Adaptive Expectations:

- Adaptive expectations:

$$E_{t-2}P_{t-1} = \frac{b}{d}\left(\frac{a-c}{b} - P_{t-1} - \frac{1}{b}u_{t-1}\right)$$

$$\Rightarrow \quad E_{t-1}P_t = \lambda P_{t-1} + (1-\lambda)\frac{b}{d}\left(\frac{a-c}{b} - P_{t-1} - \frac{1}{b}u_{t-1}\right)$$

then

$$\begin{aligned} P_t &= \frac{a-c}{b} - \frac{d}{b}\left[\lambda P_{t-1} + \frac{b(1-\lambda)}{d}\left(\frac{a-c}{b} - P_{t-1} - \frac{u_{t-1}}{b}\right)\right] - \frac{u_t}{b} \\ &= \frac{\lambda(a-c)}{b} + \frac{b-\lambda\left(b+d\right)}{b}P_{t-1} - \frac{1}{b}u_t + \frac{1-\lambda}{b}u_{t-1} \end{aligned}$$

Stationary requires

$$\left|\frac{b-\lambda\left(b+d\right)}{b}\right| < 1$$

Systematic forecast errors!

Rational Expectations:

- The idea:
  economic agents should make use of all information available to them and the efficient use of the information should result in expectations that coincide with the predictions of the model
- Two key assumptions:
  - Agents know the model, including the stochastic properties of the exogenously specified variables
  - Agents know the complete history of the economy, i.e., they can observe past realizations of all variables.
    Information set at time t

$$I_t = \{u_t, u_{t-1}, \cdots, P_{t-1}, P_{t-2}, \cdots, Q_{t-1}, Q_{t-2} \cdots\}$$

- Solutions to rational expectations models represent a class of fixed point problem: the distributions with respect to which agents take expectations are themselves part of the solution to the model

Cobweb Model Solution

- The model implies

$$\begin{aligned} E_{t-1}P_t &= \frac{a-c}{b} - \frac{d}{b}E_{t-1}E_{t-1}P_t - \frac{1}{b}E_{t-1}u_t \\ &= \frac{a-c}{b} - \frac{d}{b}E_{t-1}P_t - \frac{1}{b}E_{t-1}u_t \\ \Rightarrow & \quad E_{t-1}P_t = \frac{a-c}{b+d} - \frac{1}{b+d}E_{t-1}u_t \end{aligned}$$

- Forecast error:

$$P_t - E_{t-1}P_t = \frac{1}{b}\left(E_{t-1}u_t - u_t\right)$$

- If u_t is i.i.d $(0, \sigma^2)$, forecast error is $-u_t/b$, an i.i.d process
  If $u_t = \rho u_{t-1} + \varepsilon_t$, ε_t i.i.d $(0, \sigma^2)$ and $0 < \rho < 1$, then forecast error is $-\varepsilon_t/b$, again an i.i.d
- No systematic error!

Solving Linear RE Models

- General form:

$$AE_tY_{t+1} = BY_t + Cx_t$$

  where Y_t and x_t are $n_y \times 1$ and $n_x \times 1$ vector stochastic processes
- For a non-singular A,

$$E_tY_{t+1} = A^{-1}BY_t + A^{-1}Cx_t = MY_t + Gx_t$$

- Y_t contains two fundamentally distinct class of variables: predetermined and non-predetermined
  Non-predetermined: sometimes referred to as jump or forward-looking variables

$$Y_t = \begin{pmatrix} q_t \\ k_t \end{pmatrix}$$

  with k_t being $n_k \times 1$ ($n_k \leq n_y$)

Solving Linear RE Models

- k_t predetermined at time t if it depends only on information available at time $t-1$, for instance, capital in a economic model... q_t can be a function of any information revealed at time t. For instance, consumption in period t depends on the income of that period...
- Blanchard and Kahn (1980):

Theorem

Let n_{ue} be the number of eigenvalues of M that lie outside the unit circle.
(i) if $n_{ue} = n_y - n_k$, then for each k_0 and bounded x_t there exists a unique, bounded, Y_t satisfying the system
(ii) if $n_{ue} < n_y - n_k$, then for each k_0 and bounded x_t there exist infinitely many bounded, Y_t satisfying the system
(iii) if $n_{ue} > n_y - n_k$, then for each k_0 and bounded x_t there exists no bounded, Y_t satisfying the system

Solving Linear RE Models

- A 2×2 system:

$$E_t \begin{bmatrix} q_{t+1} \\ k_{t+1} \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} q_t \\ k_t \end{bmatrix} + \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} x_t$$

- Step 1:
  Find the eigenvalues and eigenvectors of M

$$e_i^T M = \lambda_i e_i^T$$

  where e_i is the LEFT eigenvector of M and λ_i is the corresponding eigenvalue which solves $|M - \lambda I| = P(\lambda) = 0$

$$P(\lambda) = (m_{11} - \lambda)(m_{22} - \lambda) - m_{12}m_{21}$$

Solving Linear RE Models

- Assume that $|\lambda_1| < 1$ and $|\lambda_2| > 1$
  The associated eigenvectors are obtained from

$$e_i^T \left(M - \lambda_i I\right) = 0$$

where

$$e_1 = \begin{bmatrix} d_1 \\ 1 \end{bmatrix} \text{ and } e_2 = \begin{bmatrix} d_2 \\ 1 \end{bmatrix}$$

(we can always normalize the second element to be one). Thus

$$\begin{bmatrix} d_i & 1 \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} = \lambda_i \begin{bmatrix} d_i & 1 \end{bmatrix}$$

Then

$$d_i = -\frac{m_{21}}{m_{11} - \lambda_i}$$

Solving Linear RE Models

- Step 2:
  Transform the system
  Define

$$z_t = e_2^T y_t = \begin{bmatrix} d_2 & 1 \end{bmatrix} \begin{bmatrix} q_t \\ k_t \end{bmatrix}$$

  Left-multiplying the system by e_2^T

$$\begin{aligned} E_t z_{t+1} &= e_2^T M \begin{bmatrix} q_t \\ k_t \end{bmatrix} + e_2^T G x_t \\ &= \lambda_2 e_2^T \begin{bmatrix} q_t \\ k_t \end{bmatrix} + J x_t = \lambda_2 z_t + J x_t \end{aligned}$$

  where $J = e_2^T G$

Solving Linear RE Models

- Step 3:
  Recursive forward substitution

$$E_t E_{t+1} z_{t+2} = \lambda_2 E_t z_{t+1} + J E_t x_{t+1}$$

If $E_t x_{t+j} = 0$,

$$E_t z_{t+2} = \lambda_2 E_t z_{t+1} \Rightarrow E_t z_{t+1} = \frac{1}{\lambda_2} E_t z_{t+2} = \cdots = \left(\frac{1}{\lambda_2}\right)^{j-1} E_t z_{t+j}$$

since $|\lambda_2| > 1$ and we are looking for bounded solutions,

$$\lim_{j\to\infty} \left(\frac{1}{\lambda_2}\right)^{j-1} E_t z_{t+j} = 0$$

Then

$$E_t z_{t+1} = \lambda_2 z_t + J x_t = 0$$

Solving Linear RE Models

- Step 4:
  Transform the system back

$$z_t = d_2 q_t + k_t = -\frac{1}{\lambda_2} J x_t$$

solving for q_t gives

$$q_t = -\frac{1}{d_2 k_t} - \frac{1}{d_2 \lambda_2} J x_t$$

Substitution into the original system for k_{t+1}

$$\begin{aligned} k_{t+1} &= m_{21} q_t + m_{22} k_t + a_2 x_t \\ &= \left(m_{22} - \frac{m_{21}}{d_2} \right) k_t + \left(a_2 - \frac{m_{21}}{d_2 \lambda_2} J \right) x_t \end{aligned}$$

k_t only depends on the information at time t

RBC

- A simple RBC model

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \left[\log c_t + \psi \log (1 - l_t)\right]$$

subject to

$$\begin{aligned} c_t + k_{t+1} &= k_t^{\alpha} \left(e^{z_t} l_t\right)^{1-\alpha} + (1-\delta) k_t \\ z_t &= \rho z_{t-1} + \varepsilon_t \end{aligned}$$

- Solving the model: FOCs with respect to c_t, l_t, k_{t+1}
- Log-linear approximation

  Applying linear RE solution

Thoughts on RE

- Two foundations of rational expectations equilibrium analysis
  - Optimization
  - Mutual consistency of beliefs
- Strong knowledge assumptions
  - Agents know more than the modeler

The Agenda of the short course

- Tools and conceptual issues
- Consequences of learning for the choice of macroeconomic policy

Introducing learning dynamics

- Tools and conceptual issues
- Consequences of learning for the choice of macroeconomic policy

Bounded Rationality

- Attempts to place the agent and modeler on equal footing
- Agents might proceed as:
  - Classical econometrician: know model but unsure about parameters
  - Bayesian econometrician: unsure about model and parameters but understand how they are unsure

Learning dynamics

- Analyze broad class of model that relaxes the second pillar of REE analysis
  - Retain optimization
  - Permit non-rational expectations
- Constrain the analysis by requiring the procedure to nest rational expectations

A simple Cobweb model

- Consider a variant of the Cobweb model

$$p_t = \mu + \alpha E_{t-1} p_t + \delta w_{t-1} + \eta_t \tag{1}$$

where $\mu, \alpha, \delta > 0$; w_t exogenous process; η_t a bounded iid disturbance

- Unique bounded solution:

$$p_t = \bar{a} + \bar{b} w_{t-1} + \eta_t$$

where

$$\bar{a} = (1-\alpha)^{-1} \mu, \text{ and } \bar{b} = (1-\alpha)^{-1} \delta$$

A simple Cobweb model

- Suppose agent forecast p_t as a linear function (if the agent unsure about parameters)

$$\hat{E}_{t-1}p_t = a_{t-1} + b_{t-1}w_{t-1} \tag{2}$$

where $\{a_{t-1}, b_{t-1}\}$ are currently maintained beliefs

- Price dynamics then

$$p_t = (\mu + \alpha a_{t-1}) + (\delta + \alpha b_{t-1})\, w_{t-1} + \eta_t$$

Least Squares Learning

- Beliefs $\{a_{t-1}, b_{t-1}\}$ estimates using data $\{p_i, w_i\}_{i=0}^{i=t-1}$
- Ordinary least squares (OLS) implies

$$\begin{pmatrix} a_{t-1} \\ b_{t-1} \end{pmatrix} = \left(\sum_{i=1}^{t-1} z_{i-1} z_{i-1}' \right)^{-1} \left(\sum_{i=1}^{t-1} z_{i-1} p_i \right) \tag{3}$$

where $z_i = [1 \ w_i]'$

  - Fully specified system: relations (1) to (3)

Least Squares Learning

- Under what conditions do $a_t \longrightarrow \bar{a}$ and $b_t \longrightarrow \bar{b}$ as $t \longrightarrow \infty$?

Theorem

Consider the model (1) to (3), if $\alpha < 1$ then $(a_t \; b_t) \longrightarrow (\bar{a} \; \bar{b})$ with probability one. If $\alpha > 1$, convergence occurs with probability zero

- Property $\alpha < 1$ referred to as expectational stability
- E-stability principle governed by mapping between agent beliefs and true model coefficients

Mapping

- Agents' beliefs and optimal forecast define

$$T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \mu + \alpha a \\ \delta + \alpha b \end{pmatrix}$$

  - A REE is a fixed point of this mapping
  - Application of method of undetermined coefficients

E-Stability

- Define the associated ordinary differential equation

$$\frac{d}{d\tau}\begin{pmatrix} a \\ b \end{pmatrix} = T\begin{pmatrix} a \\ b \end{pmatrix} - \begin{pmatrix} a \\ b \end{pmatrix}$$

  where τ is notional time

  - Local stability properties govern E-Stability principle

E-Stability

- In particular, the REE is E-Stable if and only if the ODE is local stable at $\left(\bar{a}\ \bar{b}\right)$
- Hence

$$\begin{aligned} \frac{da}{d\tau} &= \mu + (\alpha - 1)\, a \\ \frac{db}{d\tau} &= \delta + (\alpha - 1)\, b \end{aligned}$$

requries $\alpha < 1$

Recursive Least Squares

- OLS

$$\phi_t = \phi_{t-1} + t^{-1} R_t^{-1} z_{t-1} \left(p_t - \phi_{t-1}' z_{t-1} \right) \quad (4)$$

$$R_t = R_{t-1} + t^{-1} \left(z_{t-1} z_{t-1}' - R_{t-1} \right) \quad (5)$$

where

$$\phi_t = [a_t \; b_t]'$$

$$R_t = t^{-1} \sum_{i=1}^{t-1} z_{i-1} z_{i-1}'$$

Proof

- R_t:

$$\begin{aligned} tR_t &= (t-1)\,R_{t-1} + z_{t-1}z_{t-1}' \\ &\Rightarrow R_t = R_{t-1} + t^{-1}\left(z_{t-1}z_{t-1}' - R_{t-1}\right) \end{aligned}$$

- ϕ_t: by definition of R_t

$$\begin{aligned} \phi_t &= (tR_t)^{-1}\left(\sum_{i=1}^{t-1} z_{i-1}p_i + z_{t-1}p_t\right) \\ \phi_{t-1} &= \left((t-1)\,R_{t-1}\right)^{-1}\left(\sum_{i=1}^{t-1} z_{i-1}p_i\right) \end{aligned}$$

Then

$$\sum_{i=1}^{t-1} z_{i-1}p_i = (t-1)\,R_{t-1}\phi_{t-1} = \left(tR_t - z_{t-1}z_{t-1}'\right)\phi_{t-1}$$

Proof

- Substitute the expression of $\sum_{i=1}^{t-1} z_{i-1} p_i$ into ϕ_t expression,

$$\phi_t = (tR_t)^{-1} \left(\left(tR_t - z_{t-1} z'_{t-1} \right) \phi_{t-1} + z_{t-1} p_t \right)$$

  Hence

$$\phi_t = \phi_{t-1} + t^{-1} R_t^{-1} z_{t-1} \left(p_t - z'_{t-1} \phi_{t-1} \right)$$

- Mapping into RLS

$$\phi_t = \phi_{t-1} + t^{-1} R_t^{-1} z_{t-1} z'_{t-1} \left(T\left(\phi_{t-1}\right) - \phi_{t-1} + \eta_t \right) \quad (6)$$

$$R_t = R_{t-1} + t^{-1} \left(z_{t-1} z'_{t-1} - R_{t-1} \right) \quad (7)$$

  where

$$p_t = z'_{t-1} T\left(\phi_{t-1}\right) + \eta_t \quad (8)$$

Stochastic Approximation Methods

- Provide results characterizing convergence of such systems
  - Ljung (1977), Marcet and Sargent (1989)
- Consider stochastic recursive algorithm

$$\theta_t = \theta_{t-1} + \gamma_t Q\left(t, \theta_{t-1}, X_t\right)$$

where

$$\begin{aligned} \theta_t &: \quad \text{parameter estimates } \left(a_t, b_t, R_t\right) \\ X_t &: \quad \text{state vector (effects of } p_t\text{, } z_t \text{ and } \eta_t\text{)} \\ \gamma_t &: \quad \text{deterministic sequence of gains } \left(t^{-1}\right) \end{aligned}$$

Stochastic Approximation Methods

- Stochastic approximation approach associates the ODE

$$\frac{d\theta}{d\tau} = h\left(\theta\left(\tau\right)\right)$$

where

$$h\left(\theta\right) = \lim_{t\to\infty} EQ\left(t, \theta, X_t\right)$$

E denotes the expectation of $Q\left(t, \theta, X_t\right)$ with respect to the invariant distribution of X_t for fixed θ

- Under suitable assumptions:
  - If $\bar{\theta}$ is a locally stable equilibrium point of the ODE, then $\bar{\theta}$ is a possible point of convergence of the SRA
  - If $\bar{\theta}$ is not a locally stable equilibrium point of the ODE, then $\bar{\theta}$ is not a possible point of convergence of the SRA. That is $\theta_t \to \bar{\theta}$ with prob zero.

Example Revisted

- Define $S_{t-1} \equiv R_t$

$$\phi_t = \phi_{t-1} + t^{-1} S_{t-1}^{-1} z_{t-1} z_{t-1}' \left(T\left(\phi_{t-1}\right) - \phi_{t-1} + \eta_t \right) \tag{9}$$

$$S_t = S_{t-1} + t^{-1} \left(\frac{t}{t+1} \right) \left(z_t z_t' - S_{t-1} \right) \tag{10}$$

such that

$$\theta_t = vec\left(\phi_t, S_t\right)$$
$$X_t = \left[1 \ w_t \ w_{t-1} \ \eta_t\right]'$$
$$\gamma_t = t^{-1}$$

Example Revisted

- Hence

$$
\begin{aligned}
Q_{\phi}\left(t, \theta_{t-1}, X_t\right) &= S_{t-1} z_{t-1}\left(z_{t-1}'\left(T\left(\phi_{t-1}\right)-\phi_{t-1}\right)+\eta_t\right) \\
Q_S\left(t, \theta_{t-1}, X_t\right) &= vec\left(\left(\frac{t}{t+1}\right) z_t z_t' - S_{t-1}\right)
\end{aligned}
$$

  - To do: compute the associated ODE
  - Fix θ_t and compute expectation over X_t

Example Revisted

- Fixing (ϕ, S)

$$
\begin{aligned}
h_{\phi}(\phi, S) &= \lim_{t\to\infty} ES^{-1} z_{t-1}\left(z_{t-1}'\left(T(\phi)-\phi\right)+\eta_t\right) \\
h_S(\phi, S) &= \lim_{t\to\infty} E\left(\frac{t}{t+1}\right)\left(z_t z_t' - S\right)
\end{aligned}
$$

- Since

$$
Ez_t z_t' = Ez_{t-1} z_{t-1}' = \begin{bmatrix} 1 & 0 \\ 0 & \Omega \end{bmatrix} \equiv M
$$

where $\Omega = E\left[w_t w_t'\right]$; $E\left[z_{t-1}\eta_t\right] = 0$ and $\lim\limits_{t\to\infty} \frac{t}{t+1} = 1$

$$
\begin{aligned}
h_{\phi}(\phi, S) &= S^{-1} M \left(T(\phi)-\phi\right) \\
h_S(\phi, S) &= M - S
\end{aligned}
$$

Example Revisted

- Associated ODE

$$\begin{aligned} \frac{d\phi}{d\tau} &= S^{-1}M\left(T\left(\phi\right)-\phi\right) \\ \frac{dS}{d\tau} &= M-S \end{aligned}$$

  - The latter is globally stable for any initial $S : S \to M$
  - Therefore $S^{-1}M \to I$
- Hence

$$\frac{d\phi}{d\tau} = T\left(\phi\right) - \phi$$

  - Intuition? Learning converges to REE

Beliefs

- Under RE:
  1. Agents optimize given beliefs
  2. The probabilities they assign to future state variables coincide with the predictions of the model
- Learning:

  Retains (1) and replaces (2) with

  2' Future state variables outside agent's control are forecasted using an econometric model

Knowledge

- Know own preferences and constraints
- Do not know true economic model of determination of variables outside their control
  - Uncertain about the equilibrium mechanisms determining prices and policy variables
  - E.g. even if $Y_t^i = Y_t^j = Y_t$ in equilibrium $\nRightarrow$ agents know $Y_t^i = Y_t^j$
- Observe aggregate variables and disturbances
- Forecast variables outside their control using an atheoretical VAR
  - Takes the minimum state variable form

DSGE models

- Agents
  - Households
  - Firms
  - Monetary authority
  - Fiscal authority

Eusepi and Preston (2010)

- A theory of expectations-driven business cycles based on learning
- Agents have incomplete knowledge about how market prices are determined and shifts in expectations of future prices affect dynamics
- The theoretical framework amplifies and propagates technology shocks

Eusepi and Preston (2010)

- Model setup

  See Appendix

ALM

- Constructing ALM (actual law of motion): Consider a model that can be cast in this form

$$
\begin{aligned}
A_0 Y_t \quad = \quad & A_c + \tilde{A}_1 \hat{E}_{t-1} Y_t + \sum_{s=1}^{k_Y} \tilde{A}_{1+s} \hat{E}_t Y_{t+s} \\
& + \sum_{s=1}^{j_Y} \tilde{A}_{k_Y+2+s} \left(\hat{E}_t \sum_{T=t}^{\infty} \gamma_s^{T-t} Y_{T+1} \right) + L Y_{t-1} \\
& + B_0 X_{t-1} + B_1 X_t + \sum_{s=1}^{k_X} B_{1+s} \hat{E}_t X_{t+s} \\
& + \sum_{s=1}^{j_X} B_{k_X+2+s} \left(\hat{E}_t \sum_{T=t}^{\infty} \gamma_s^{T-t} X_{T+1} \right) + C \epsilon_t
\end{aligned}
$$

and shocks hit exogenous variables

$$X_t = H X_{t-1} + \epsilon_t^s$$

ALM

- Re-write the model in compact form, as described by the following matrices A_i, $i = 0...k + j + 3$,

$$\begin{bmatrix} A_0 & 0 \\ 0 & I \end{bmatrix} Z_t = A_1 + A_2 \hat{E}_{t-1} Z_t + \sum_{s=1}^{k} A_{s+2} \hat{E}_t Z_{t+s} + \sum_{s=1}^{j} A_{s+k+2} \left(\hat{E}_t \sum_{T=t}^{\infty} \gamma_s^{T-t} Z_{T+1} \right) + A_{j+k+3} Z_{t-1} + C\epsilon_t$$

where k denotes the maximum (finite) forecasting horizon in the model and j denotes the number of different discount rate appearing in the infinite horizon expressions,

$$A_1 = \begin{bmatrix} A_c \\ 0 \end{bmatrix}$$

ALM

- Variables are **ordered** as follows (this is important for the solution and simulation files)
  - the first variables are not state variables (do not enter in the PLM, perceived law of motion)
  - last variables are the exogenous shocks

PLM

- The PLM can be expressed as

$$Y_t = \Omega_0 + \Omega_Y Y_{t-1} + \Omega_X X_{t-1} + \Omega_\epsilon \epsilon_t$$

but we can re-write it in compact notation

$$Z_t = \begin{bmatrix} Y_t \\ X_t \end{bmatrix} = \Omega_{z,0} + \Omega_Z Z_{t-1} + \Omega_{z,\epsilon}\epsilon_t$$

where Z_t is of dimension n_{XY}. Under the assumption that the agents estimate the MSV solution and know the matrix H

$$\Omega_Z = \begin{bmatrix} 0_{n\_e \times n\_e} & \tilde{\Omega}_Z \\ 0_{n\_s \times n\_s} & H \end{bmatrix}$$

When estimating parameter values, Y_{t-1} is a linear projection of X_{t-1}

PLM

- **Expectations** can be computed as follows

$$\hat{E}_t Z_{T+1} = \left(I_{n_{XY}} - \Omega_Z\right)^{-1} \left(I_{n_{XY}} - \Omega_Z^{T-t+1}\right) \Omega_{z,0} + \Omega_Z^{T-t+1} Z_t$$

For $T \geq t$. Concerning the infinite sums we have that

$$\hat{E}_t \sum_{T=t}^{\infty} \gamma_s^{T-t} Z_{T+1} = F_0^s + F_1^s Z_t$$

where

$$F_0^s = \left(I_{n_{XY}} - \Omega_Z\right)^{-1} \left[\left(1-\gamma_s\right)^{-1} I_{n_{XY}} - \Omega_Z \left(I_{n_{XY}} - \gamma_s \Omega_Z\right)^{-1}\right] \Omega_{z,0}$$

$$F_1^s = \Omega_Z \left(I_{n_{XY}} - \gamma_s \Omega_Z\right)^{-1}$$

PLM

- Then

$$Z_t = T_0(\Omega_{z,0}, \Omega_Z) + T_c(\Omega_{z,0}, \Omega_Z)Z_t + T_L(\Omega_{z,0}, \Omega_Z)Z_{t-1} + C\epsilon_t$$

$$\begin{aligned} T_0(\Omega_{z,0}, \Omega_Z) &= A_0^{-1}A_1 + A_0^{-1}A_2\Omega_{z,0} \\ &+ A_0^{-1}\sum_{s=1}^{k} A_{2+s}\left(I_{n_{XY}} - \Omega_Z\right)^{-1}\left(I_{n_{XY}} - \Omega_Z^s\right)\Omega_{z,0} \\ &+ A_0^{-1}\sum_{s=1}^{j} A_{s+k+2}F_0^s \end{aligned}$$

$$T_c(\Omega_{z,0}, \Omega_Z) = A_0^{-1}\sum_{s=1}^{k} A_{2+s}\Omega_Z^s + A_0^{-1}\sum_{s=1}^{j} A_{s+k+2}F_1^s$$

$$T_L(\Omega_{z,0}, \Omega_Z) = A_0^{-1}A_2\Omega_Z + A_0^{-1}A_{j+k+3}$$

PLM

- Finally, ALM becomes

$$
\begin{aligned}
Z_t \quad = \quad & (I_{n_{XY}} - T_c(\Omega_{z,0}, \Omega_Z))^{-1} T_0(\Omega_{z,0}, \Omega_Z) \\
& + (I_{n_{XY}} - T_c(\Omega_{z,0}, \Omega_Z))^{-1} T_L(\Omega_{z,0}, \Omega_Z) Z_{t-1} \\
& + (I_{n_{XY}} - T_c(\Omega_{z,0}, \Omega_Z))^{-1} C\epsilon_t
\end{aligned}
$$

This result can be completely different from REE!

Households

- Households seek to maximize

$$\hat{E}_t^i \sum_{T=t}^{\infty} \beta^{T-t} \left[U\left(C_T^i; \xi_T\right) - v\left(h_T^i; \xi_T\right)\right]$$

subject to

$$\begin{aligned} B_t^i &= (1 + i_{t-1}) B_{t-1}^i + w_t h_t^i + \int_0^1 \Pi_t(j)\, dj - T_t - P_t C_t^i \\ &= (1 + i_{t-1}) B_{t-1}^i + P_t Y_t^i - T_t - P_t C_t^i \end{aligned}$$

  - Assume agents forecast period income

FOCs

- Log-linear approximation yields

$$\hat{C}_t^i = \hat{E}_t^i \hat{C}_{t+1}^i - \sigma \left(i_t - \hat{E}_t^i \pi_{t+1} - g_t + \hat{E}_t^i g_{t+1} \right)$$

and

$$\hat{E}_t^i \sum_{T=t}^{\infty} \beta^{T-t} \hat{C}_T^i = A_t^i + \hat{E}_t^i \sum_{T=t}^{\infty} \beta^{T-t} \hat{Y}_T^i$$

where

$$A_t^i = B_t^i / \left(P_t \bar{Y} \right); g_t = \xi_t u_{c\xi} / u_c; \sigma^{-1} = -u_{cc} \bar{Y} / u_c$$

  - Assume government debt in zero net supply $\int_i B_t^i di = 0$.

Consumption decision rule

- Solving the Euler equation backwards recursively from T to t

$$\hat{E}_t^i \hat{C}_T^i = \hat{C}_t^i - g_t + \hat{E}_t^i \left[g_T + \sigma \sum_{T=t}^{\infty} (i_t - \pi_{t+1}) \right]$$

  Use this to eliminate future expected consumption in the intertemporal budget constraint

- Then

$$\begin{aligned} \hat{C}_t^i &= (1-\beta) A_t^i \\ &\quad + \hat{E}_t^i \sum_{T=t}^{\infty} \beta^{T-t} \left[(1-\beta) \hat{Y}_T^i - \beta\sigma (i_t - \pi_{t+1}) + \beta (g_T - g_{T+1}) \right] \end{aligned}$$

  - An example of permanent income theory
  - Forecasts about prices into the indefinite future matter!
  - A particular important source of uncertainty are anticipated future interest rates!

Conparison to Existing Approaches

- Are decisions implied by the optimal decision equivalent to those implied by

$$\hat{C}_t^i = \hat{E}_t^i \hat{C}_{t+1}^i - \sigma \left(i_t - \hat{E}_t^i \pi_{t+1} - g_t + \hat{E}_t^i g_{t+1} \right)$$

  - where forecasts determined by the model

$$z_t^* = a_{t-1} + b_{t-1} z_{t-1} + \varepsilon_t$$

where $z_t^* = \left[\hat{C}_t^i \ \pi_t \ i_t \ g_t \right]'$ and $z_t = \left[\hat{Y}_t \ \pi_t \ i_t \ g_t \right]'$

- In general, NO!

Forecasting Consumption

- The optimal forecast

$$\begin{aligned}\hat{C}_t^i &= (1-\beta) A_t^i \\ &+ \hat{E}_t^i \sum_{T=t}^{\infty} \beta^{T-t} \left[(1-\beta) \hat{Y}_T^i - \beta\sigma \left(i_t - \pi_{t+1} \right) + \beta \left(g_T - g_{T+1} \right) \right]\end{aligned}$$

- Not the same as

$$\hat{E}_t^i z_{t+1}^* = a_{t-1} + b_{t-1} z_t$$

Aggregate Consumption Dynamics

- Aggregating over the continuum provides

$$\hat{C}_t = \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left[(1-\beta) \hat{Y}_T - \beta\sigma \left(i_t - \pi_{t+1} \right) + \beta \left(g_T - g_{T+1} \right) \right]$$

where

$$\hat{C}_t \equiv \int \hat{C}_t^i di; \hat{E} \equiv \int \hat{E}_t^i di; 0 \equiv \int A_t^i di$$

  - For any variable X_t, $\hat{E}_t \hat{E}_{t+1} X_{t+j} \neq \hat{E}_t X_{t+j}$

Summary

- Under RE, expectations are consistent over time
- Under learning dynamics expectations about future endogenous decision variables are taken with respect to a distribution
  - Induced by beliefs about exogenous states
  - Induced by optimal decisions

Firms

- Continuum of firms maximize

$$\hat{E}_t^j \sum_{T=t}^{\infty} \alpha^{T-t} Q_{t,T} \Pi_T^j \left(p_t\left(j\right)\right)$$

where

$$\begin{aligned} \Pi_t^j\left(p\right) &= Y_T P_T^{\theta} p^{1-\theta} - w_T^i f^{-1}\left(Y_T P_T^{\theta} p^{-\theta} / A_T\right) \\ y_t^i &= A_t f\left(h_t^i\right) \end{aligned}$$

Optimal Price Setting

- Log-linear approximation implies

$$p_t^j = \hat{E}_t^j \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \left[\frac{(1-\alpha\beta)\left(\omega+\sigma^{-1}\right)}{1+\omega\theta} x_T + \pi_T \right]$$

where

$$x_T = Y_t - \hat{Y}_t^n$$

is the output gap

Model Summary

- Aggregate demand and supply:

$$
\begin{aligned}
x_t &= \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left[(1-\beta) x_{T+1} - \sigma \left(i_T - \pi_{T+1} - r_T \right) \right] \\
\pi_t &= \hat{E}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \left[\kappa x_T + (1-\alpha) \beta \pi_{T+1} + u_T \right]
\end{aligned}
$$

where

$$
\begin{aligned}
r_t &= \rho r_{t-1} + \varepsilon_t \text{ and } 0 < \rho < 1 \\
u_t &= \gamma u_{t-1} + \eta_t \text{ and } 0 < \gamma < 1
\end{aligned}
$$

Refer to the notes on monetary economics

Rational Expectation

- An important feature of RE is that we have

$$\hat{E}_t \hat{E}_{t+1} X_{t+1} = \hat{E}_t X_{t+1}$$

Hence

$$\begin{aligned} x_t &= \hat{E}_t \left[(1-\beta) x_{t+1} - \sigma (i_t - \pi_{t+1} - r_t) \right] \\ &\quad + \beta \hat{E}_t \sum_{T=t+1}^{\infty} \beta^{T-(t+1)} \left[(1-\beta) x_{T+1} - \sigma (i_T - \pi_{T+1} - r_T) \right] \\ &= \hat{E}_t \left[(1-\beta) x_{t+1} - \sigma (i_t - \pi_{t+1} - r_t) \right] + \beta \hat{E}_t x_{t+1} \\ &= E_t x_{t+1} - \sigma \left(i_t - \hat{E}_t \pi_{t+1} - r_t \right) \end{aligned}$$

Rational Expectation

- Similarly

$$\begin{aligned}
\pi_t &= \kappa x_t + (1-\alpha)\,\beta\hat{E}_t\pi_{t+1} \\
&\quad + (\alpha\beta)\,\hat{E}_t \sum_{T=t+1}^{\infty} (\alpha\beta)^{T-(t+1)} \left[\kappa x_T + (1-\alpha)\,\beta\pi_{T+1}\right] \\
&= \kappa x_t + (1-\alpha)\,\beta\hat{E}_t\pi_{t+1} + \alpha\beta\hat{E}_t\pi_{t+1} \\
&= \kappa x_t + \beta\hat{E}_t\pi_{t+1}
\end{aligned}$$

EGP (2019): On the Limits of Monetary Policy

- Theory: distorted long-term interest-rate expectations represent a fundamental constraint on monetary policy design
- Optimal policy: less aggressive relative to RE case
  - More aggressive policy leads to sub-optimal volatility in long-term interest rates and aggregate demand

RE Model

- Under RE:
  - Divine Coincidence: Optimal policy can completely stabilize inflation and the output gap when faced with movements in the natural real rate of interest
- However
  - The power of monetary policy rests on the assumption that long-term expectations are anchored
  - If this pre-condition is not met, there are fundamental limits to what monetary policy can do

Co-movement btw Forecasts and Realized Variables

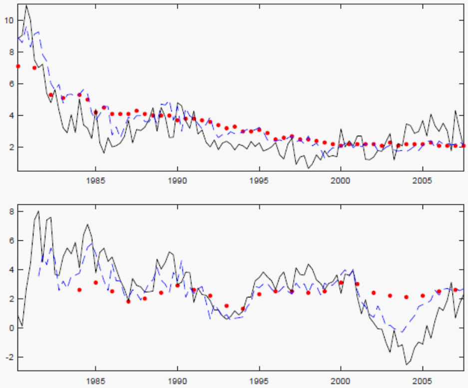

Figure 1: **Long-run expectations**

The how the evolution of survey expectations data for inflation (top panel) and the short-term ex-post real rate of interest (bottom panel). Realized variables are shown in solid black lines, four-quarters ahead forecast in dashed blue lines and five-to-ten years ahead forecasts are in red dots. Source: Blue Chip Economics and Survey of Professional Forecasters.

A Learning Model

- Refer to EGP (2019): On the Limits of Monetary Policy

Beliefs

- Log-linear approximation to optimal decisions and market clearing conditions

$$A_0 z_t = \sum_{s=1}^{3} A_s \hat{E}_t \sum_{T=t}^{\infty} \lambda_s^{-(T-t)} z_{T+1} + A_4 z_{t-1} + A_5 \varepsilon_t \tag{11}$$

z_t is the vector collects all model variables, the vector ε_t collects exogenous innovations. λ_s are the model's unstable eigenvalues

  - This representation holds for arbitrary beliefs

Beliefs

- Each agent has a common forecasting model

$$
\begin{aligned}
z_t &= S\bar{\omega}_t + \Phi z_{t-1} + e_t \\
\bar{\omega}_t &= \rho\bar{\omega}_{t-1} + u_t
\end{aligned}
$$

  - $0 < \rho < 1$, e_t and u_t IID with $R = E\left[e_t e_t'\right]$ and $Q = E\left[u_t u_t'\right]$
  - $\bar{\omega}_t$ is an unobserved state, capturing imperfect knowledge about the conditional mean of z_t
  - RE: $\bar{\omega}_t = \bar{\omega}_{t-1} = 0$ when $Q = 0$, that is, the prior belief about the variance-covariance matrix of the drift terms is zero

Expectation

- The forecasting model implies conditional expectation

$$E_t z_{t+n} = \Phi^n z_t + \sum_{j=0}^{n} \Phi^j S \rho^{n-j} \bar{\omega}_t$$

  - The first term: conventional auto-regressive impact of the current state
  - The second term: effects of drifting beliefs on conditional expectations
  - Which is more important? To interpret the data pattern (on the forecast of interest rate), we need either highly persistent shocks, or highly persistent changes in beliefs.
  - In the case $\rho = 1$

$$\lim_{n\to\infty} E_t z_{t+n} = (I - \Phi)^{-1} S \bar{\omega}_t$$

Objective Beliefs

- Given an estimate of ω, we can evaluate expectations required for optimal decisions as

$$E_t \sum_{T=t}^{\infty} \lambda_s^{-(T-t)} z_{T+1} = F_0(\lambda_s) S\omega_t + F_1(\lambda_s) z_t$$

(11) becomes

$$\begin{aligned} z_t &= \left(A_0 - \sum_{j=1}^{3} A_s F_1(\lambda_s)\right)^{-1} \left[\sum_{j=1}^{3} A_s F_0(\lambda_s) S\omega_t + A_4 z_{t-1} + A_5 \varepsilon_t\right] \\ &= T(\Phi^*) S\omega_t + \Phi^* z_{t-1} + \Phi_\varepsilon^* \varepsilon_t \end{aligned}$$

where

$$\Phi^* \equiv \left(A_0 - \sum_{j=1}^{3} A_s F_1(\lambda_s)\right)^{-1} A_4, \Phi_\varepsilon^* \equiv \left(A_0 - \sum_{j=1}^{3} A_s F_1(\lambda_s)\right)^{-1} A_5$$

Subjective Belief Updating

- Updating

$$
\begin{aligned}
\omega_{t+1} &= \rho\omega_t + P_t S' \left(S P_t S' + R\right)^{-1} F_t \\
P_{t+1} &= P_t - P_t S' \left(P_t + R\right)^{-1} S P_t' + Q
\end{aligned}
$$

where

$$
F_t \equiv z_t - S\omega_t - \Phi^* z_{t-1}
$$

P_t is the mean square error associated with the estimate ω_t

Subjective Belief Updating

- To derive Kalman filter equations

$$\hat{\omega}_{t+1} = \rho\hat{\omega}_t + K_t\left(z_t - S\hat{\omega}_t - \Phi z_{t-1}\right)$$

where K_t is Kalman gain

$$\omega_{t+1} - \hat{\omega}_{t+1} = \rho\left(\omega_t - \hat{\omega}_t\right) - K_t\left(S\omega_t - S\hat{\omega}_t + e_t\right) + u_t$$

Then

$$\begin{aligned} P_{t+1} &= E\left[\left(\omega_{t+1} - \hat{\omega}_{t+1}\right)\left(\omega_{t+1} - \hat{\omega}_{t+1}\right)'\right] \\ &= \left(\rho I - K_t S\right) P_t \left(\rho I - K_t S\right)' + K_t R K_t' + Q \end{aligned}$$

Subjective Belief Updating

- Minimize trace of P_t

$$\frac{dTr\left(P_t\right)}{dK_t} = 0 \Rightarrow K_t = \rho P_t S' \left(SP_t S' + R\right)^{-1}$$

where we used the fact that

$$\frac{dTr\left(AXB\right)}{dX} = \left(BA\right)'$$

Then we obtain the recursive forms

Subjective Belief Updating

- Following Sargent and Williams (2005), re-scale the posterior estimate using $P_t = \Xi_t R$, use the approximation $(I + \Xi_t)^{-1} \simeq I$ for small Ξ_t

$$\begin{aligned} \omega_{t+1} &= \rho\omega_t + \rho\Xi_t S' F_t \\ \Xi_{t+1} &= \rho^2 \Xi_t - \rho^2 \Xi_t \Xi_t + QR^{-1} \end{aligned}$$

  - Steady state Ξ must be something like $\Xi = \alpha I$ (we right multiply both sides of the second equation by Ξ^{-1} in the steady state), and $Q = g^2 R$ as a restriction
  - Then

$$\begin{aligned} \omega_{t+1} &= \rho\omega_t + \rho\alpha S' \left(z_t - S\omega_t - \Phi^* z_{t-1}\right) \\ &= \left[\rho + \rho\alpha S' \left(T\left(\Phi^*\right) - I\right) S\right] \omega_t + \rho\alpha S' \Phi^* \varepsilon_t \end{aligned}$$

State-space Representation

- Linear state-space representation

$$Z_t = F(\Theta) Z_{t-1} + Q(\Theta) \varepsilon_t$$

where Θ defines the set of model parameters

$$F(\Theta) = \begin{bmatrix} \Phi^* & T(\Phi^*) S\rho & T(\Phi^*) S\rho\alpha \\ 0 & \rho I & \rho\alpha I \\ 0 & S' [T(\Phi^*) - I] S\rho & S' [T(\Phi^*) - I] S\alpha \end{bmatrix}$$

and

$$Z_t = \begin{bmatrix} z_t \\ \omega_t \\ S' F_t \end{bmatrix} \text{ and } Q(\Theta) = \begin{bmatrix} \Phi_\varepsilon^* \\ 0 \\ S' \Phi_\varepsilon^* \end{bmatrix}$$

State-space Representation

- The last equation comes from

$$
\begin{aligned}
S'F_t &= S'z_t - S'S\omega_t - S'\Phi^* z_{t-1} \\
&= S'\left(T\left(\Phi^*\right)S\omega_t + \Phi^* z_{t-1} + \Phi^*_\varepsilon \varepsilon_t\right) - S'S\omega_t - S'\Phi^* z_{t-1} \\
&= S'\left(T\left(\Phi^*\right)S\left(\rho\omega_{t-1} + \rho\alpha S'F_{t-1}\right) + \Phi^*_\varepsilon \varepsilon_t\right) - S'S\left(\rho\omega_{t-1} + \rho\Xi_t\right. \\
&= S'\left(T(\Phi^* - I\right)\rho\omega_{t-1} + S'\left[T\left(\Phi^*\right) - I\right]S\rho\alpha S'F_{t-1} + S'\Phi^*_\varepsilon \varepsilon_t
\end{aligned}
$$

Policy

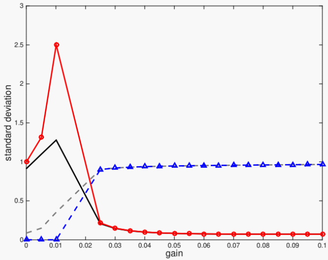

Figure 10: **Volatility as a function of the constant gain.**

This figure show the volatility of output and interest rates as a function of the constant gain $\bar{g}$. The welfare theoretic loss gives the volatility of the interest rate (red circles) and the output gap (blue triangles); while a policy maker with a concern of interest rate volatility delivers the interest rate shown by the black line, and the output gap given by the grey dashed line.

A Simple Case

- Consider the loss function facing central bank

$$L_t = \pi_t^2 + \lambda_x x_t^2$$

subject to

$$\begin{aligned} x_t &= \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left[(1-\beta) x_{T+1} - \left(R_T - \pi_{T+1} - r_T^n \right) \right] \\ \pi_t &= \hat{E}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \left[\kappa x_T + (1-\alpha) \beta \pi_{T+1} \right] \end{aligned}$$

we consider log utility

A Simple Case

- Agents have a forecasting model

$$z_t = \begin{bmatrix} \pi_t \\ x_t \\ R_t \end{bmatrix} \text{ and } \omega_t = \begin{bmatrix} \omega_t^{\pi} \\ \omega_t^{x} \\ \omega_t^{R} \end{bmatrix}$$

- Central bank has rational expectations and complete information
- iid shocks to r_t^n, private agent beliefs initially consistent with RE $\omega_{t-1} = 0$. In this case

$$\pi_t = \kappa x_t \text{ and } x_t = -\left(R_t - r_t^n\right)$$

where $E_t z_T = 0$ for $T > t$ based on initial forecast. The complete stabilization requires

$$R_t = r_t^n,\ \pi_t = x_t = 0$$

A Simple Case

- Movements in beliefs

$$\omega_t^R = \omega_{t-1}^R + g\left(R_t - \omega_{t-1}^R\right) = \omega_{t-1}^R + g\left(r_t^n - \omega_{t-1}^R\right)$$

That is $E_{t+1} R_T = \omega_t^R$ for $T > t+1$.

Next period stabilization problem

$$\begin{aligned} \pi_{t+1} &= \kappa x_{t+1} \\ x_{t+1} &= -\left(R_{t+1} - r_{t+1}^n\right) - \frac{\beta}{1-\beta}\omega_t^R \end{aligned}$$

Complete stabilization implies

$$R_{t+1} = r_{t+1}^n - \frac{\beta}{1-\beta}\omega_t^R$$

Optimal interest rate need not only respond to shocks in natural rate, but also movements in long run interest rates driven by expectations

A Simple Case

- Movements in beliefs

$$\begin{aligned} \omega_{t+1}^{R} &= \omega_{t}^{R} + g\left(R_{t+1} - \omega_{t}^{R}\right) \\ &= \left(1 - \frac{g}{1-\beta}\right)\omega_{t}^{R} + g r_{t+1}^{n} \end{aligned}$$

- Dynamics of beliefs need to be stationary, note that $g > 0$

$$1 - \frac{g}{1-\beta} > -1 \Rightarrow g < 2\left(1-\beta\right)$$

For larger gains, stability is not feasible, implying beliefs and interest rates are explosive. This is not a permissible, or at least desirable